Thermodynamics

of the

Quark-Gluon Plasma

Claudia Ratti

Torino University and INFN, Italy

Part II

QCD thermodynamics

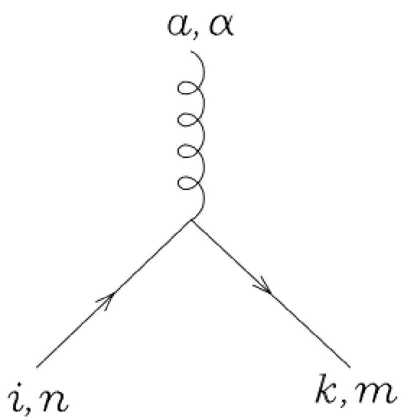
in the perturbative

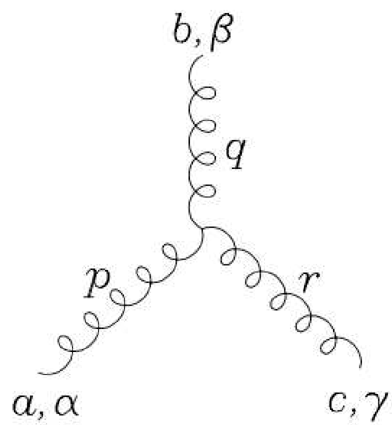
and non-perturbative regimes

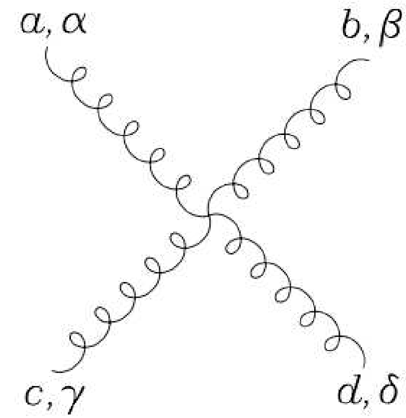
QCD is very complicated...

Let us have a look at the QCD lagrangian: $\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + \bar{\psi}^r \left(i \not{D} - m_r \right) \psi^r$,
where:

$$F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$







- Nonperturbative solution of QCD is terribly complicated

- Perturbation theory can be applied when α_s is small

  ➡ $\alpha_s(Q^2) = \frac{12\pi}{(33-2n_f)\log\left(\frac{Q^2}{\Lambda^2}\right)}$ $\quad Q \sim T$ $\quad$ applicable at large T

- Lattice QCD: well-established non-perturbative approach to solve QCD

Stefan-Boltzmann limit

- Simplest possible system: non-interacting gas of massless quarks and gluons

- we expect QCD thermodynamics to reach this limit at $T \to \infty$

$$n = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\mathrm{e}^{p/T} \pm 1} = \nu \frac{\zeta(3)}{\pi^2} T^3 \qquad \nu = \begin{cases} 1 & bosons \\ \frac{3}{4} & fermions \end{cases}$$

where $\zeta(3) = 1.202$ (Riemann ζ function)

$$\epsilon = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{p}{\mathrm{e}^{p/T} \pm 1} = \nu' \frac{\pi^2}{30} T^4 \qquad \nu' = \begin{cases} 1 & bosons \\ \frac{7}{8} & fermions \end{cases}$$

pressure: $p = \dfrac{\epsilon}{3}$ entropy density: $Ts = \epsilon + P = \dfrac{4}{3}\epsilon \quad \Longrightarrow \quad s = \dfrac{4}{3}\dfrac{\epsilon}{T} = 2\nu' \dfrac{\pi^2}{45} T^3$

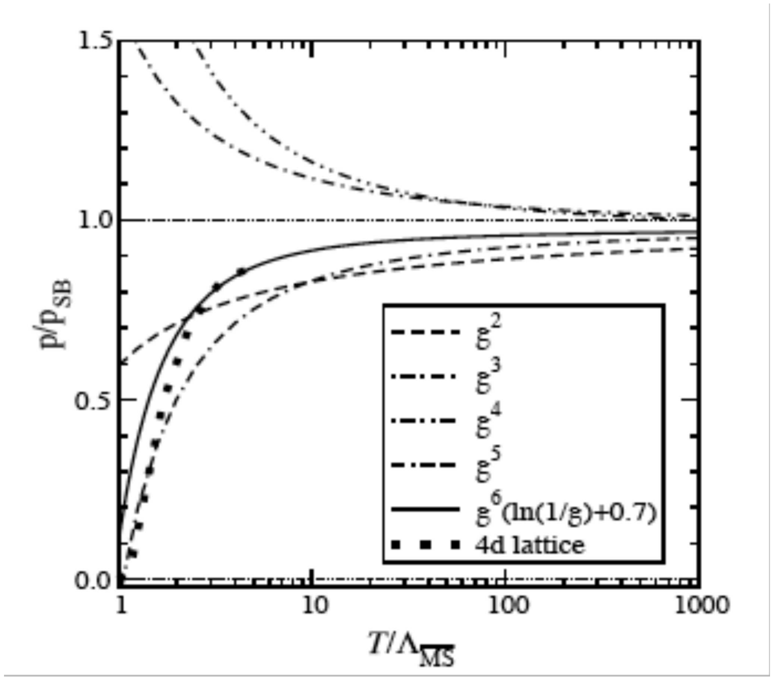
$\dfrac{n}{T^3}, \dfrac{p}{T^4}, \dfrac{s}{T^3}, \dfrac{\epsilon}{T^4}$ constant in SB limit!

Comparison to Stefan-Boltzmann limit

- Comparison to SB limit very useful
- information on how strong is the interaction in my system
- plots always show SB limit
- or alternatively, rescale thermodynamic quantities in plots by SB value: $p/p_{SB},\ \epsilon/\epsilon_{SB},\ s/s_{SB}$...
- in SB limit we have $\epsilon = 3p$

  ➠ $\dfrac{\epsilon - 3p}{T^4}$ interaction measure or trace anomaly

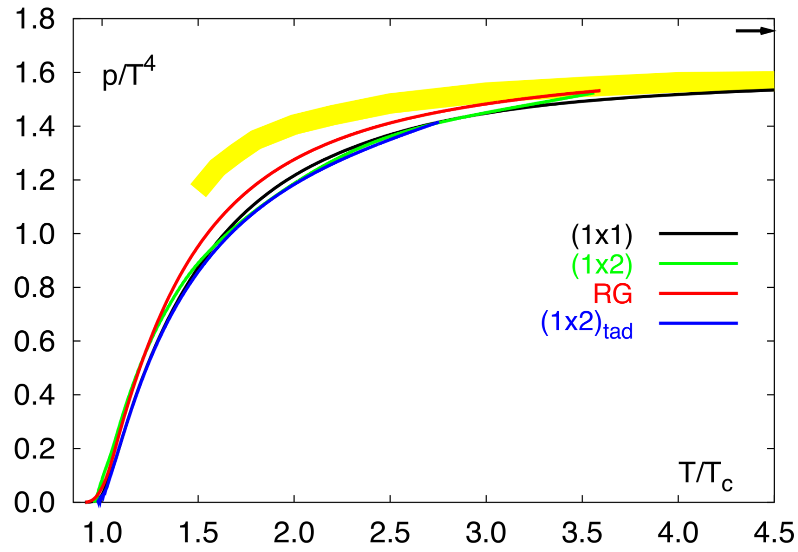
Switch on interaction: perturbative QCD



- Much effort put into calculating the successive orders of the perturbative expansion for the pressure
- the series is known now up to order $g^6 \log g$
- perturbation theory makes sense only for very small values of the coupling constant
- For not too small values of the coupling, the successive terms in the expansion oscillate

F. Kajantie et al, PRL86, PRD67

Improve series convergence



J.P. Blaizot, E. Iancu, A. Rebhan, PLB470

- One can improve the convergence of the series by some clever resummation
- Hard Thermal Loop: Quark and Gluon propagators are dressed by some effective mass
- this improves the series convergence and the agreement to lattice data down to $T \sim 3T_c$

(See lecture by A. Beraudo)

Lattice QCD (I)

- Lattice QCD: well-established non-perturbative approach to solving QCD
- formulated on a grid or lattice of points in space and time
- no new parameters or field variables are introduced in this discretization
  - ➡ LQCD retains the fundamental character of QCD
- the discrete space-time lattice acts as a non-perturbative regularization scheme
  - ➡ At finite values of the lattice spacing a, which provides an ultraviolet cutoff at π/a, there are no infinities
  - ➡ renormalized physical quantities have a finite well behaved limit as $a \to 0$
- LQCD can be simulated on the computer using methods analogous to those used for Statistical Mechanics systems

Lattice QCD (II)

- non-perturbative implementation of field theory using the Feynman path integral approach

$$Z = \int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S}$$

- Fermions can be integrated out exactly: (M: Dirac operator)

$$Z = \int \mathcal{D}A_\mu \det M e^{-\int d^4x \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right)}$$

- After this integration, the action can be written as:

$$S = S_{gauge} + S_{quarks} = \int d^4x \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right) - \sum_i \log(\det M_i)$$

- Calculation of expectation values

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}A_\mu \mathcal{O} e^{-S}$$

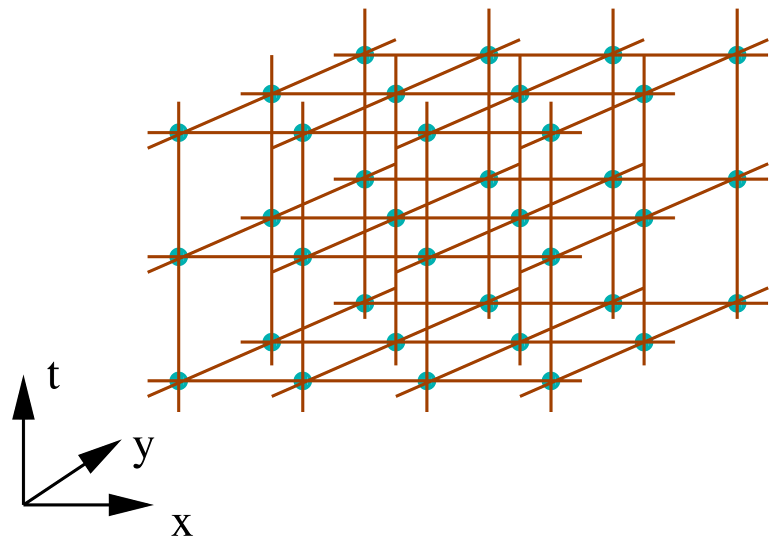
Lattice QCD (III)

The numerical implementation of the path integral approach requires the following five steps:

- Discretization of space-time
- Transcription of the gauge and fermion degrees of freedom
- Construction of the action
- Definition of the measure of integration in the path integral
- Transcription of the operators used to probe the physics

Discretization of space-time

- Simplest: isotropic hypercubic grid with spacing $a = a_S = a_T$ and size $N_S \times N_S \times N_S \times N_T$.



- Physical size of the lattice: $L = N_S a$
- Temperature: $T = \dfrac{1}{N_T a}$
- N_T large $\Rightarrow$ a small: closer to continuum limit but computationally expensive

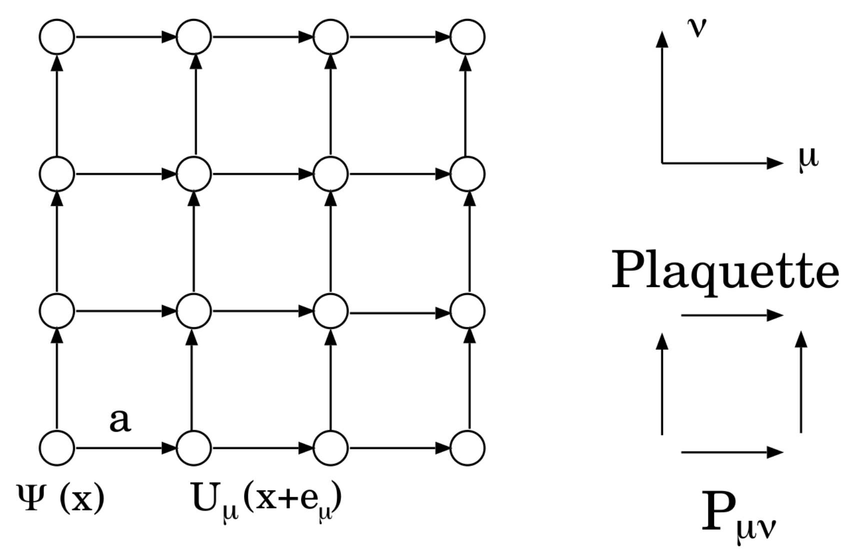
Transcription of the gauge and fermion degrees of freedom

- The quark field is represented by anticommuting Grassmann variables defined at each site of the lattice
- In the continuum, the gauge fields $A_\mu(x)$ carry 4-vector Lorentz indices, and mediate interactions between fermions
- a fermion moving from site x to y in presence of a gauge field $A_\mu(x)$ picks up a phase factor given by the path ordered product

$$\psi(y) = \mathcal{P} e^{\int_x^y igA_\mu(x) dx_\mu} \psi(x)$$

- Gauge fields are associated with links that connect sites on the lattice

$$U_\mu(x) = U(x, x+\hat{\mu}) = e^{iagA_\mu(x+\frac{\hat{\mu}}{2})}$$



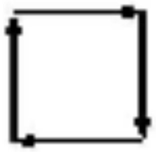
Fundamental fields

Lattice gauge action

- The gauge action can be expressed in terms of closed loops

- Example: abelian $U(1)$ model

$$W_{\mu\nu} = U_\mu(x)U_\nu(x+\hat{\mu})U^\dagger_\mu(x+\hat{\nu})U^\dagger_\nu(x)$$



Expanding it about $x + \frac{\hat{\mu}+\hat{\nu}}{2}$ gives:

$$Re\mathrm{Tr}(1 - W_{\mu\nu}) = \frac{a^4 g^2}{2} F_{\mu\nu}F^{\mu\nu} \quad \Rightarrow \quad \frac{1}{g^2}\sum_x \sum_{\mu<\nu} Re\mathrm{Tr}(1 - W_{\mu\nu}) = \frac{a^4}{2}\sum_x \sum_{\mu,\nu} F_{\mu\nu}F^{\mu\nu}$$

to lowest order in a, the expansion of the plaquette gives the continuum action!

- gauge action for $SU(3)$:

$$S_g = \frac{6}{g^2}\sum_x \sum_{\mu<\nu} Re\mathrm{Tr}\frac{1}{3}(1 - W_{\mu\nu})$$

Lattice fermion action

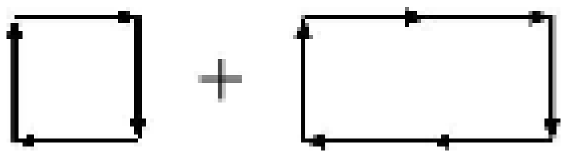
- replace the derivative with the symmetrized difference and include appropriate gauge links to maintain gauge invariance

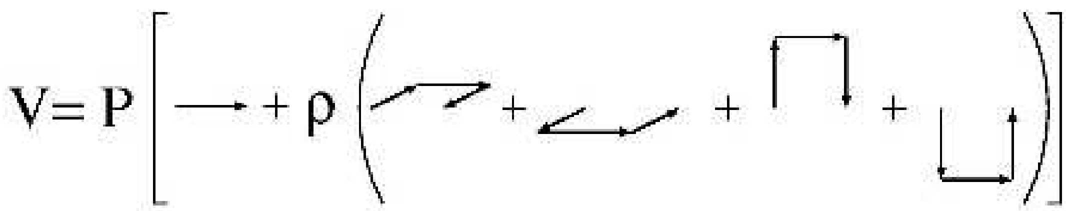
$$\bar{\psi}\not{D}\psi = \frac{1}{2a}\bar{\psi}\sum_{\mu}\gamma_{\mu}\left[U_{\mu}(x)\psi(x+\hat{\mu}) - U^{\dagger}_{\mu}(x-\hat{\mu})\psi(x-\hat{\mu})\right]$$

Keeping only the leading term in a one arrives at the simplest (called naive) lattice action for fermions :

$$S^{N} = m_q\sum_{x}\bar{\psi}\psi + \frac{1}{2a}\sum_{x}\bar{\psi}(x)\gamma_{\mu}\left[U_{\mu}(x)\psi(x+\hat{\mu}) - U^{\dagger}_{\mu}(x-\hat{\mu})\psi(x-\hat{\mu})\right]$$

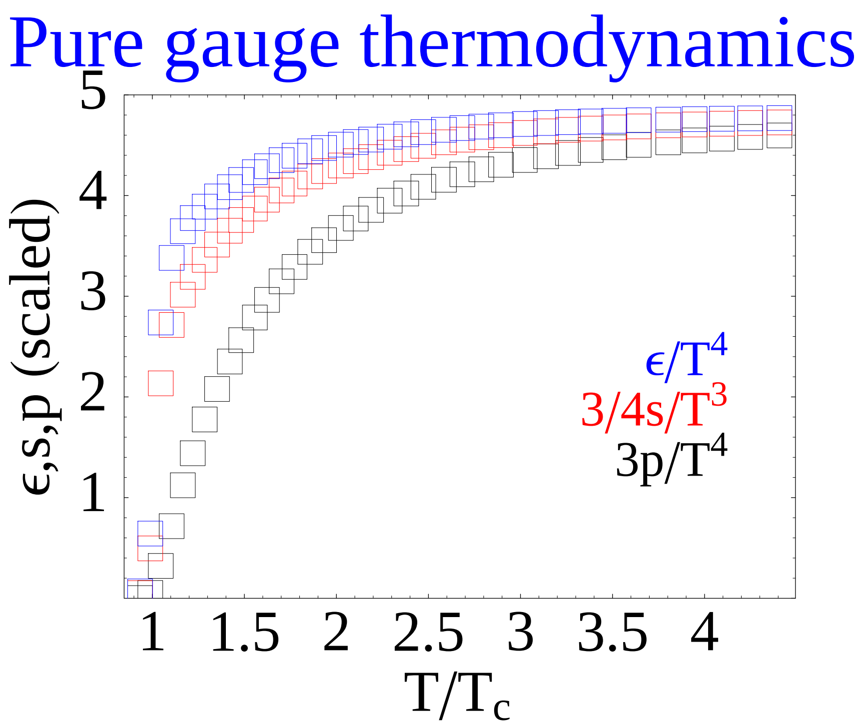
- improvement for gauge and fermionic actions: reduce discretization effects

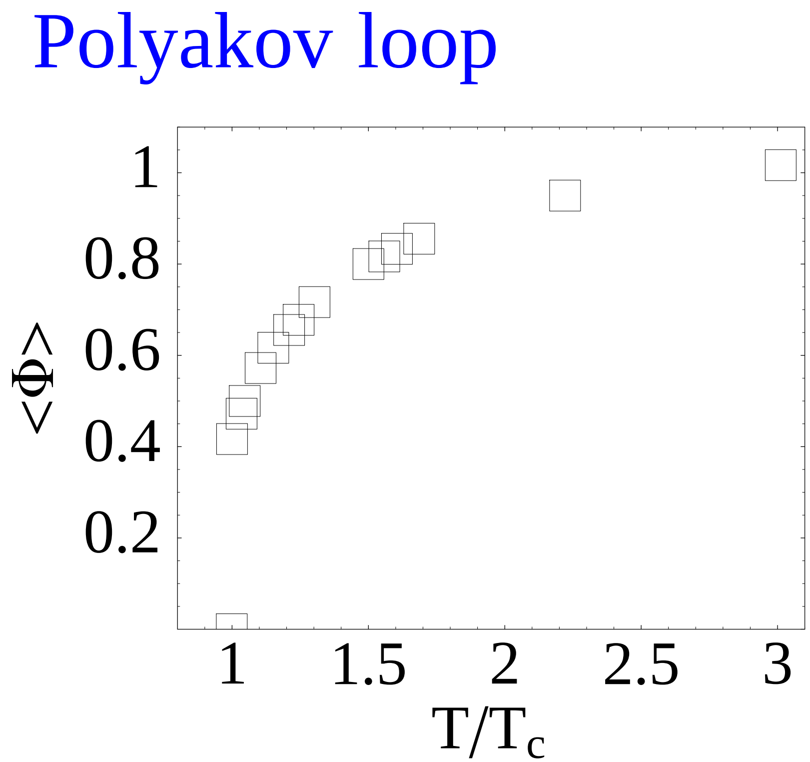




- input: $g,\ m_q$; scale setting: a: fixed by measuring physical quantities ($m_K,\ f_K, m_{\pi}$)

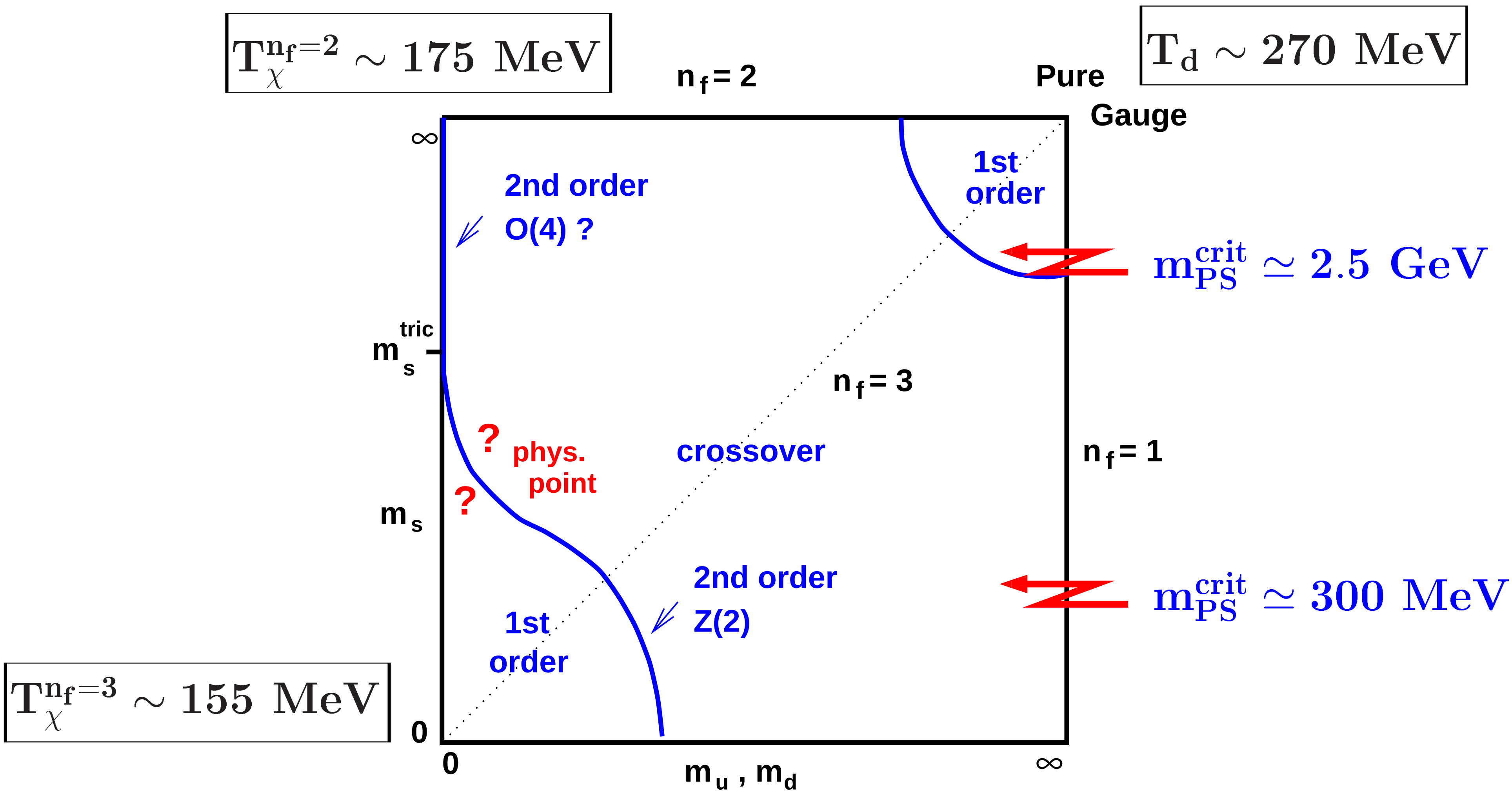
Pure gauge lattice QCD results



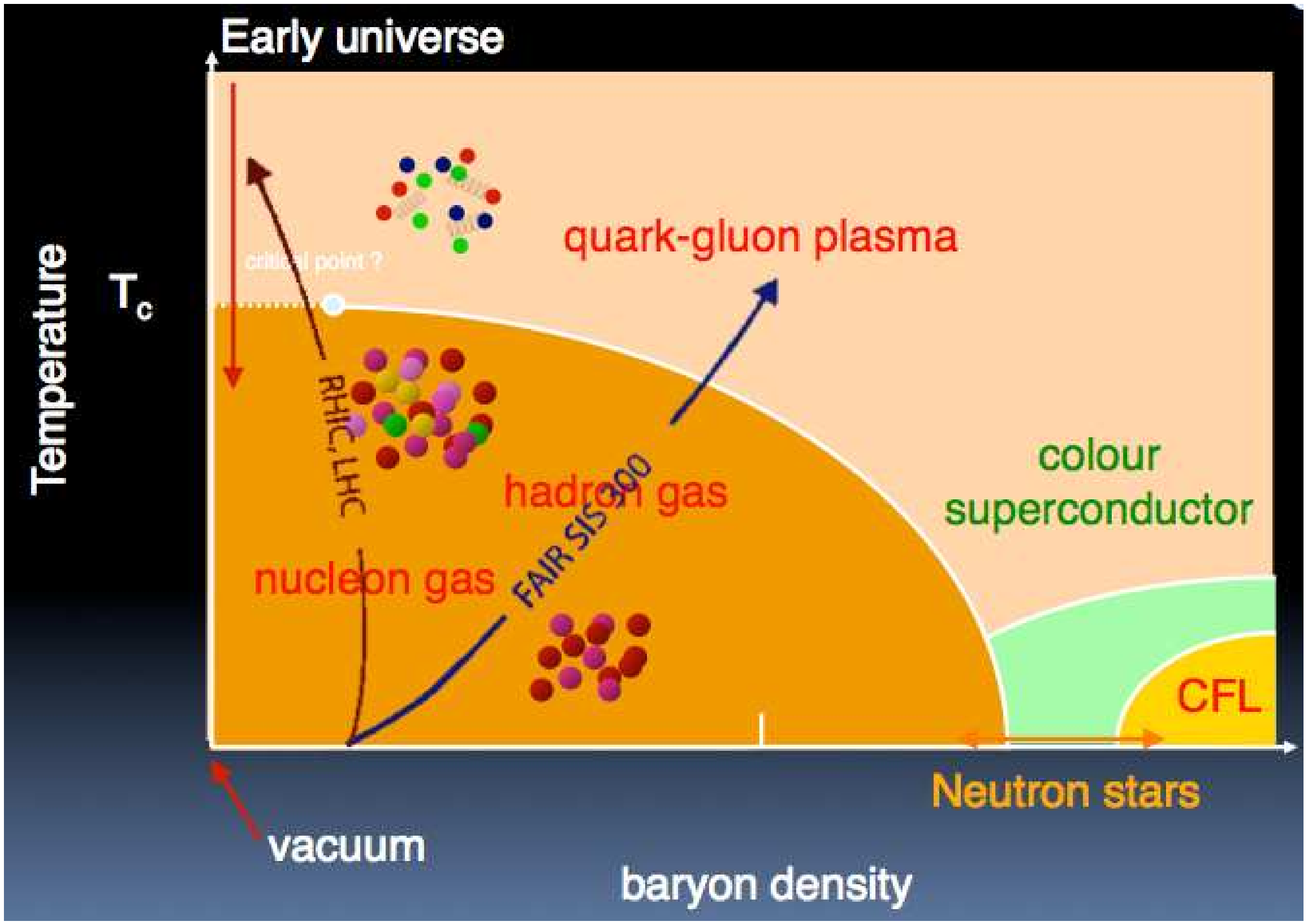


- data for thermodynamic quantities (Boyd et al. NPB469(1996)) and Polyakov loop (Kaczmarek et al. PLB543(2002))
- discontinuity: first order phase transition
- transition temperature in pure gauge: 270 MeV

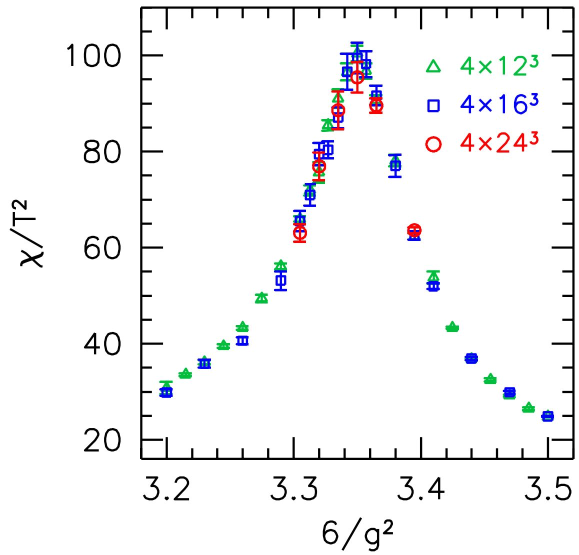
Phase diagram and quark masses

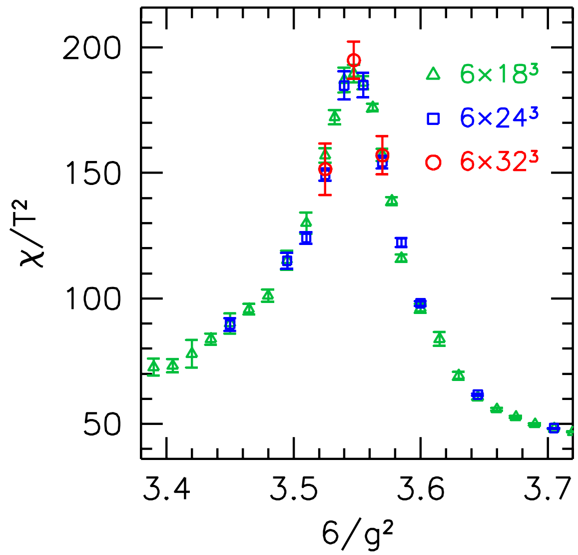


The QCD phase diagram



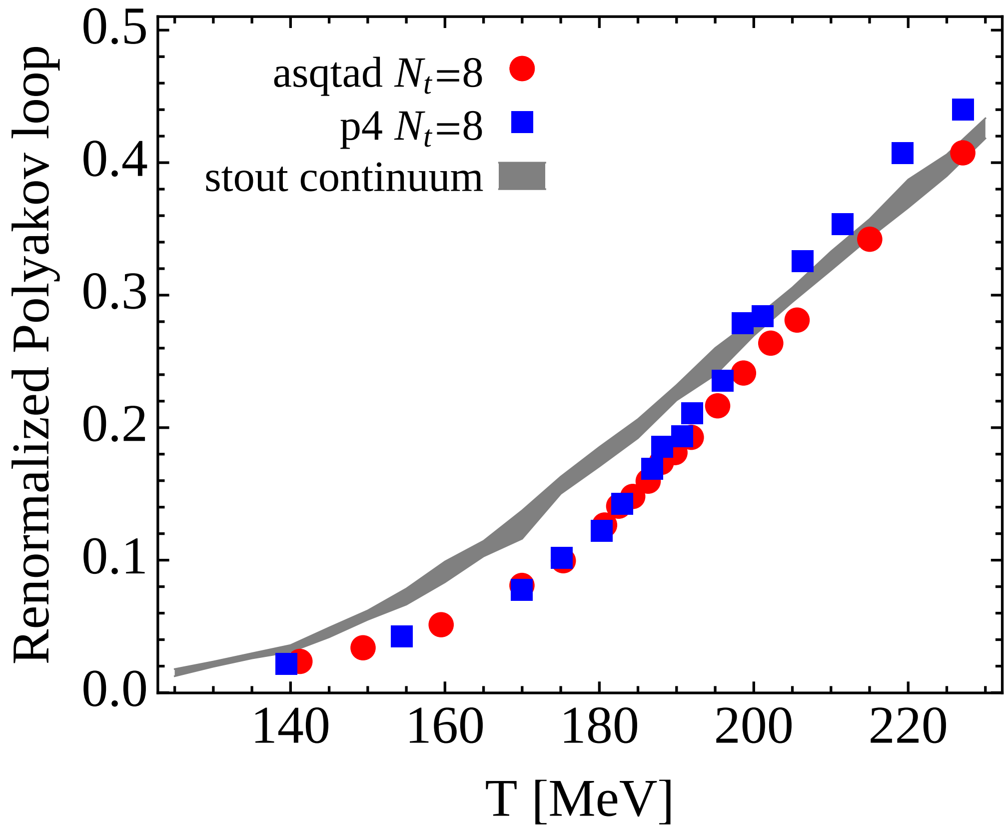
Order of the phase transition





Aoky *et al.*, Nature 443 (2006):*"The largest volume is eight times bigger than the smallest one, so a first-order phase transition would predict a susceptibility peak that is eight times higher (for a second-order phase transition the increase would be somewhat less, but still dramatic). Instead of such a significant change we do not observe any volume dependence."*

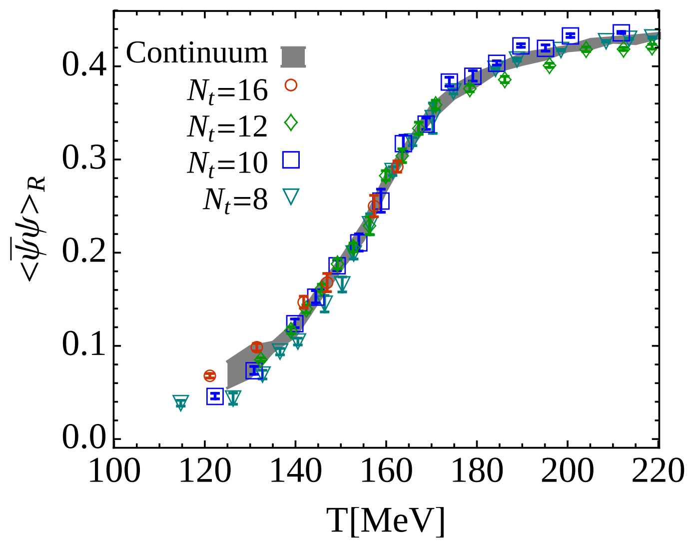
Results: Polyakov loop

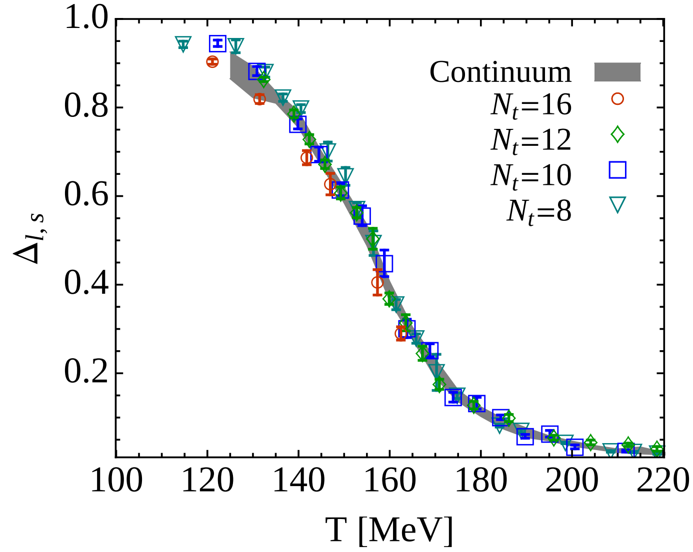


- Polyakov loop changes very smoothly with T (Borsanyi et al. JHEP1009 (2010))

Results: chiral condensate

$$\langle\bar{\psi}\psi\rangle_R = -\left[\langle\bar{\psi}\psi\rangle_{l,T} - \langle\bar{\psi}\psi\rangle_{l,0}\right]\frac{m_l}{X^4} \quad \text{with} \quad \langle\bar{\psi}\psi\rangle_l = \frac{T}{V}\frac{\partial \ln Z}{\partial m_l}$$





$$\Delta_{l,s} = \frac{\langle\bar{\psi}\psi\rangle_{l,T} - \frac{m_l}{m_s}\langle\bar{\psi}\psi\rangle_{s,T}}{\langle\bar{\psi}\psi\rangle_{l,0} - \frac{m_l}{m_s}\langle\bar{\psi}\psi\rangle_{s,0}}$$

- Chiral condensate changes very smoothly with T (Borsanyi et al. JHEP1009 (2010))

Equation of state: integral method

- On the lattice, the dimensionless pressure is given by:

$$p^{lat}(\beta, m_q) = (N_t N_s^3)^{-1} \log Z(\beta, m_q)$$

- only its derivatives are accessible using conventional algorithms:

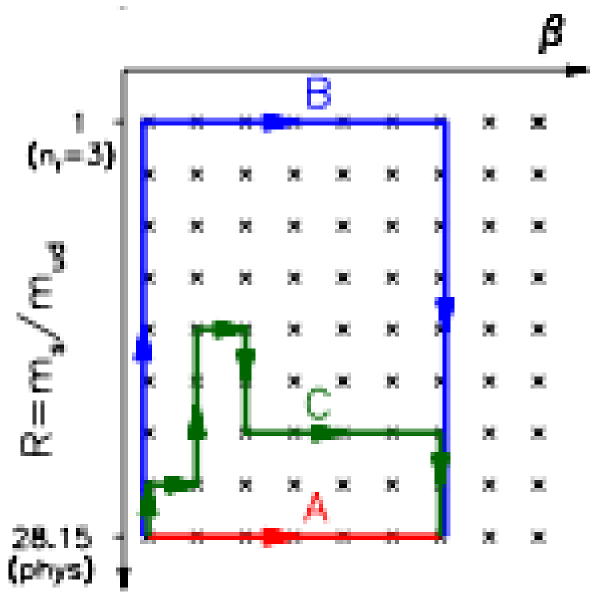
$$p^{lat}(\beta, m_q) - p^{lat}(\beta^0, m_q^0) = (N_t N_s^3)^{-1} \int_{(\beta^0, m_q^0)}^{(\beta, m_q)} \left[d\beta \frac{\partial \log Z}{\partial \beta} + dm_q \frac{\partial \log Z}{\partial m_q} \right]$$

- the pressure has to be renormalized: subtraction at $T = 0$ (or $T > 0$)

- $T \neq 0$ simulations cannot go below $T \simeq 100$ MeV (lattice spacing is large)

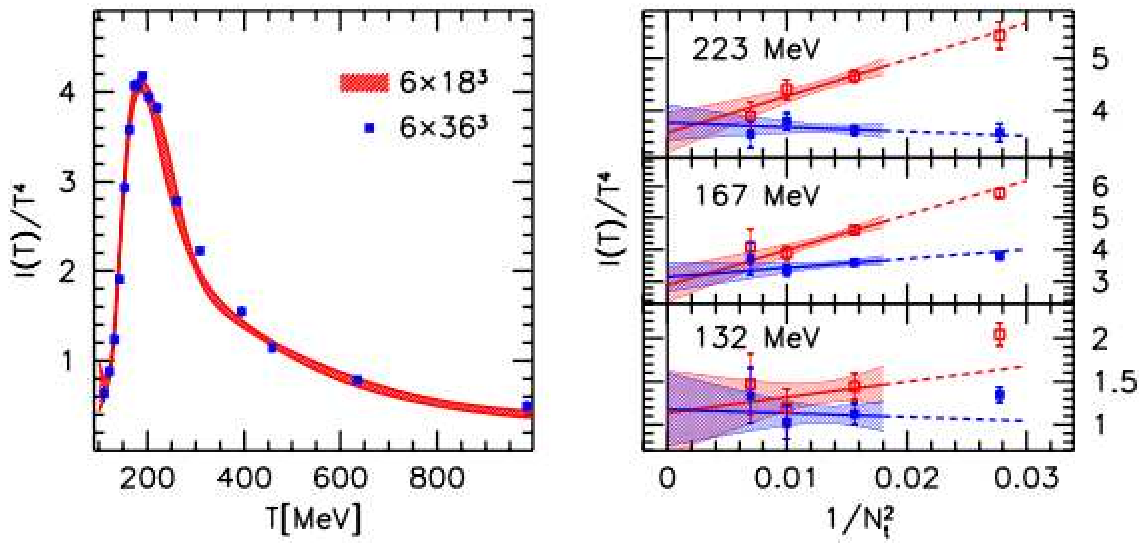
All path approach

- Our goal:
  - determine the equation of state for several pion masses
  - reduce the uncertainty related to the choice of β^0



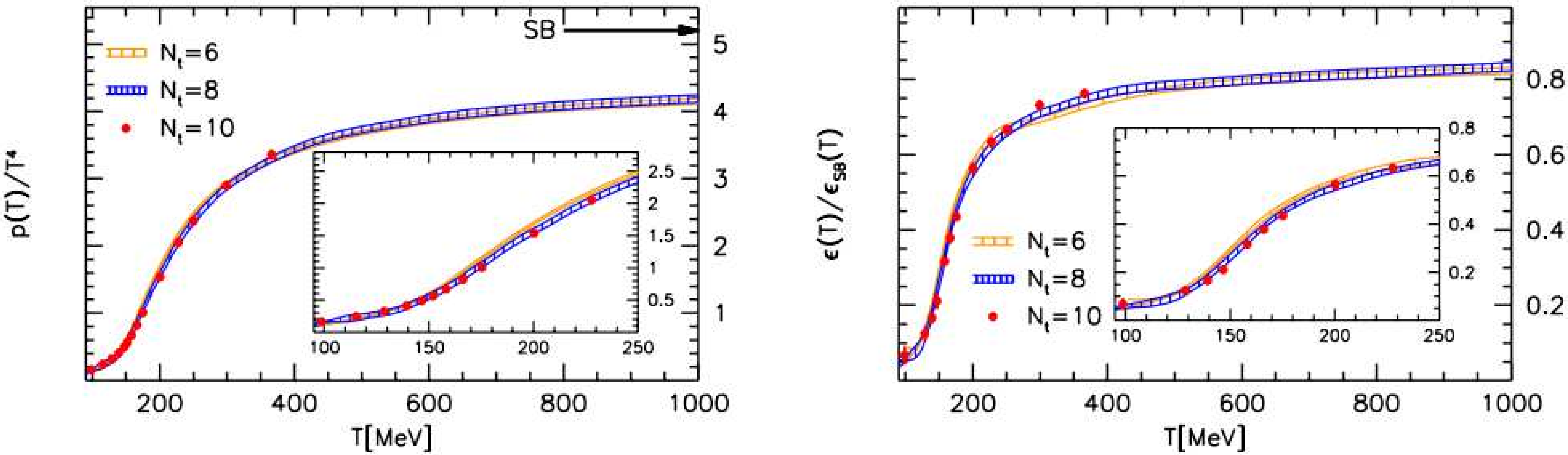
- conventional path: A, though B, C or any other paths are possible
- generalize: take all paths into account

Finite volume and discretization effects



- finite V : $N_s/N_t = 3$ and 6 (8 times larger volume): no sizable difference
- finite a: improvement program of lattice QCD (action observables)
  - tree-level improvement for p (thermodynamic relations fix the others)
  - trace anomaly for three T-s: high T, transition T, low T
  - continuum limit $N_t = 6, 8, 10, 12$: same with or without improvement
- improvement strongly reduces cutoff effects: slope$\simeq 0$ ($1 - 2\sigma$ level)

Results: pressure and energy density

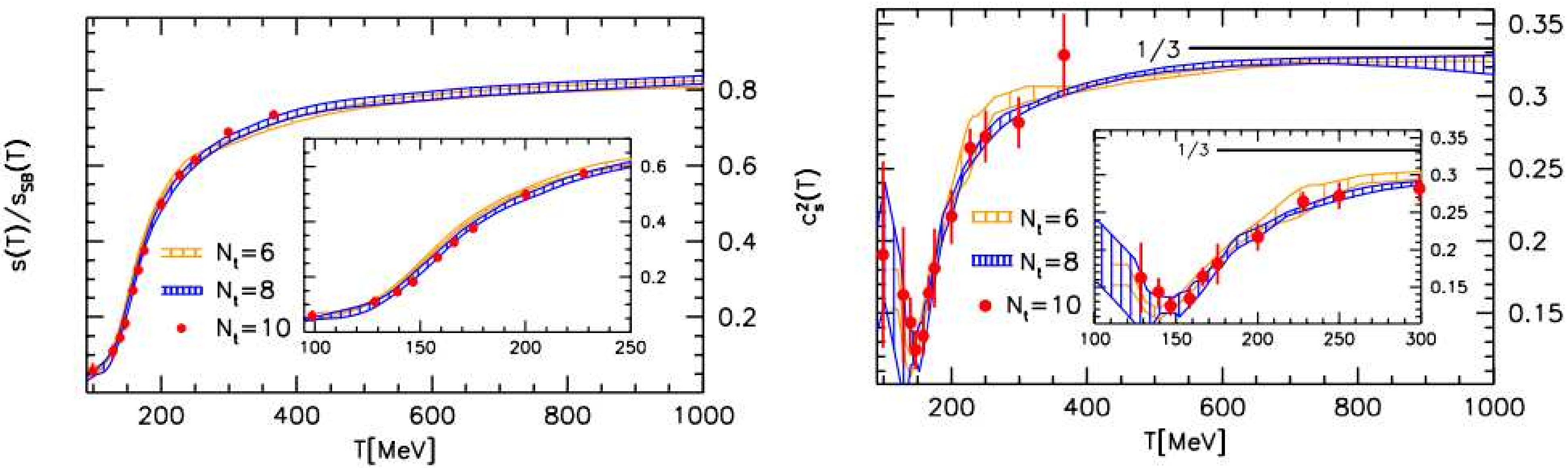


- The different N_t data are on top of each other
- The energy density is rescaled by the SB limit $\epsilon_{SB}/T^4 = 15.7$
- At T$\simeq 1000$ MeV these quantities reach $\sim$ 80% of the SB limit

S. Borsanyi *et al.*, JHEP1011 (2010)

Results: entropy and speed of sound

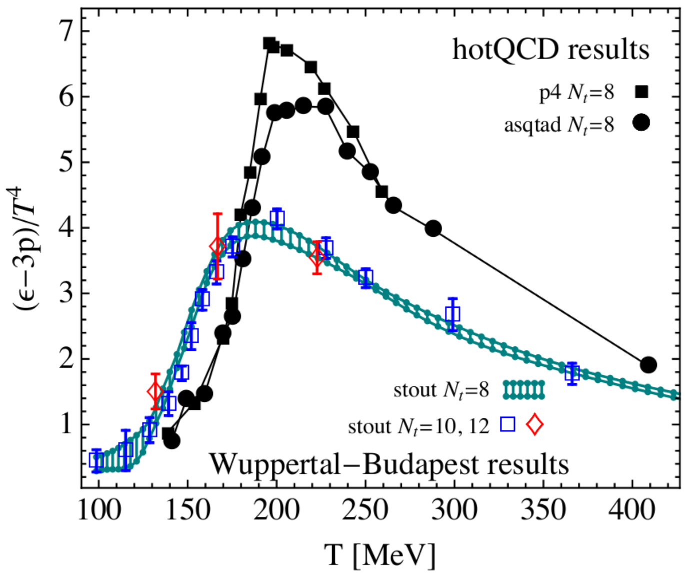
$$\boxed{c_s^2 = \frac{dp}{d\epsilon}}$$

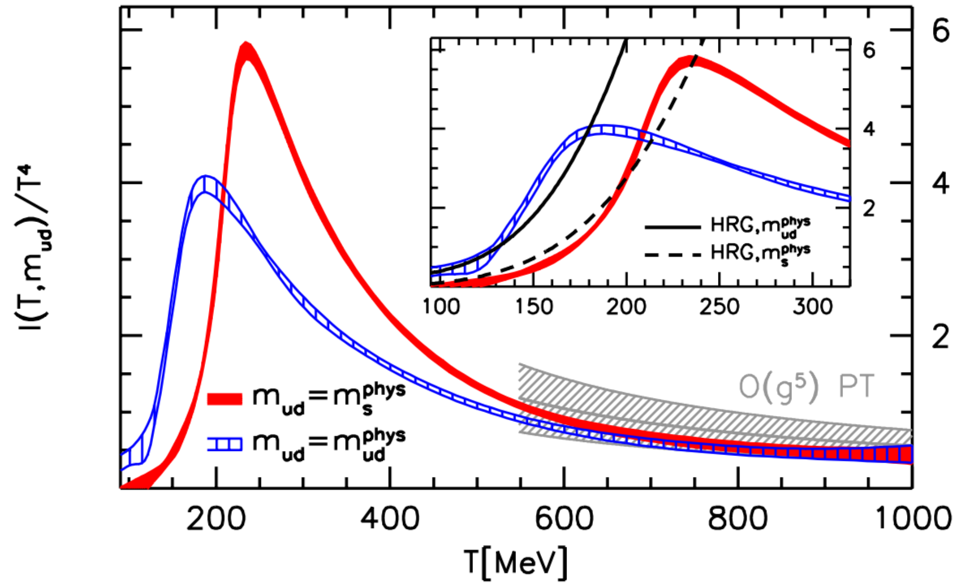


- The different N_t data are on top of each other
- The entropy is rescaled by the SB limit
- c_s^2 minimum value is about 0.13 at $T \simeq 145$ MeV

S. Borsanyi *et al.*, JHEP1011 (2010)

Trace anomaly





- comparison with the published results of the hotQCD collaboration
  - discrepancy: peak at $\simeq 20$ MeV larger T and $\simeq 50\%$ higher
- two different pion masses: $M_\pi = 135$ MeV and $M_\pi \simeq 720$ MeV
- good agreement with the HRG model up to the transition region
- quark mass dependence disappears for high T
- good agreement with perturbation theory

T_c summary from Wuppertal-Budapest collaboration

	$\chi_{\bar{\psi}\psi}/T^4$	$\Delta_{l,s}$	$\langle\bar{\psi}\psi\rangle_R$	χ_2^s/T^2	ϵ/T^4	$(\epsilon-3p)/T^4$
WB'10	147(2)(3)	157(3)(3)	155(3)(3)	165(5)(3)	157(4)(3)	154(4)(3)
WB'09	146(2)(3)	155(2)(3)	-	169(3)(3)	-	-
WB'06	151(3)(3)	-	-	175(2)(4)	-	-

- Different variables give different T_c values: the transition is broad

S. Borsanyi *et al.*, JHEP1009 (2010)

Summary of part II

- Perturbation theory works relatively well at large T

  ➠ non-perturbative methods are needed

- lattice QCD: well-established non-perturbative method to solve QCD

- put quarks and gluons on a discretized grid

- build action for gluons and quarks

- thermodynamics at $\mu = 0$

- predictions for T_c