Fourth Order RK-Method

The most commonly used method is Runge-Kutta fourth order method.

The fourth order RK-method is

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where

$$\begin{aligned} k_1 &= hf(x_i, y_i), \\ k_2 &= hf\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right), \\ k_3 &= hf\left(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}\right), \\ k_4 &= hf(x_i + h, y_i + k_3). \end{aligned}$$

Example 1

Find the approximate solution of the initial value problem

$$\frac{dx}{dt} = 1 + \frac{x}{t}, \quad 1 \leq t \leq 3$$

with the initial condition

$$x(1) = 1,$$

using the Runge-Kutta second order and fourth order with step size of $h = 1$.

Solution

RK 2nd order method. The formula is

$$y_{i+1} = y_i + \frac{h}{2}\left(k_1 + k_2\right),$$

where $k_1 = f(x_i, t_i),\ k_2 = f(x_i + h, t_i + hk_1)$.

Here, $h = 1$ and $t_0 = x_0 = 1$. Therefore,

$$k_1 = f(t_0, x_0) = 1 + \frac{x_0}{t_0} = 2,$$

$$k_2 = f(t_0 + h, x_0 + hk_1) = f(2, 3) = 1 + \frac{3}{2} = 2.5.$$

Thus,

$$y_1 = 1 + \frac{1}{2}(2 + 2.5) = 3.25.$$

Similary, we calculate y_2.

RK 4th order method. The formula is

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where

$$\begin{aligned}
k_1 &= hf(t_i, x_i), \\
k_2 &= hf\left(t_i + \frac{h}{2}, x_i + \frac{k_1}{2}\right), \\
k_3 &= hf\left(t_i + \frac{h}{2}, x_i + \frac{k_2}{2}\right), \\
k_4 &= hf(t_i + h, x_i + k_3).
\end{aligned}$$

$$
\begin{aligned}
k_1 &= hf(t_0, x_0) = f(1,1) = 2, \\
k_2 &= hf\left(t_0 + \frac{h}{2}, x_0 + \frac{k_1}{2}\right) = f(1.5, 2) = 2.333333, \\
k_3 &= hf\left(t_0 + \frac{h}{2}, x_0 + \frac{k_2}{2}\right) = 2.444444, \\
k_4 &= hf(t_0 + h, x_0 + k_3) = 2.722222.
\end{aligned}
$$

We find

$$\begin{aligned} y_1 &= 1 + \frac{1}{6}(2 + 2 * 2.333333 \\ &+ 2 * 2.444444 + 2.722222) \\ &= 3.379630. \end{aligned}$$

Similarly, we calculate y_2.

System of First Order ODE's

We consider the n system of first order equations

$$
\begin{aligned}
\frac{dy_1}{dx} &= f_1(x, y_1, y_2, \cdots, y_n),\ y_1(x_0) = y_{10},\\
\frac{dy_2}{dx} &= f_2(x, y_1, y_2, \cdots, y_n),\ y_2(x_0) = y_{20},\\
&\cdots\\
\frac{dy_n}{dx} &= f_n(x, y_1, y_2, \cdots, y_n),\ y_n(x_0) = y_{n0}.
\end{aligned}
$$

With the help of single step methods, we find the approximate solution for the system of n equations.

We now consider the two first order equations of the form

$$
\begin{aligned}
\frac{dy_1}{dx} &= f_1(x, y_1, y_2),\ y_1(x_0) = y_{10}, \\
\frac{dy_2}{dx} &= f_2(x, y_1, y_2),\ y_2(x_0) = y_{20}.
\end{aligned}
$$

We use RK second order and fourth order method to solve the system of equations.

2nd Order RK Method

The formula for second order RK method with step size h is

$$\begin{aligned} y_{1,n+1} &= = y_{1,n} + \frac{1}{2}(s_1 + k_1), \\ y_{2,n+1} &= y_{2,n} + \frac{1}{2}(s_2 + k_2), \end{aligned}$$

where

$$\begin{aligned} k_1 &= hf_1(x_n, y_{1n}, y_{2n}), \\ k_2 &= hf_2(x_n, y_{1n}, y_{2n}), \end{aligned}$$

$$\begin{aligned} s_1 &= hf_1(x_n + h, y_{1n} + k_1, y_{2n} + k_2), \\ s_2 &= hf_2(x_n + h, y_{1n} + k_1, y_{2n} + k_2). \end{aligned}$$

Similarly, the formula for fourth order RK-method is

$$\begin{aligned} y_{1,n+1} &= y_{1,n} + \frac{1}{6}(k_1 + 2s_1 + 2l_1 + p_1), \\ y_{2,n+1} &= y_{2,n} + \frac{1}{6}(k_2 + 2s_2 + 2l_2 + p_2), \end{aligned}$$

where

$$
\begin{aligned}
k_1 &= hf_1(x_n, y_{1n}, y_{2n}), \\
k_2 &= hf_2(x_n, y_{1n}, y_{2n}), \\
s_1 &= hf_1(x_n + \frac{h}{2}, y_{1n} + \frac{k_1}{2}, y_{2n} + \frac{k_2}{2}), \\
s_2 &= hf_2(x_n + \frac{h}{2}, y_{1n} + \frac{k_1}{2}, y_{2n} + \frac{k_2}{2})
\end{aligned}
$$

$$
\begin{aligned}
l_1 &= hf_1(x_n + \frac{h}{2}, y_{1n} + \frac{s_1}{2}, y_{2n} + \frac{s_2}{2}), \\
l_2 &= hf_2(x_n + \frac{h}{2}, y_{1n} + \frac{s_1}{2}, y_{2n} + \frac{s_2}{2}), \\
p_1 &= hf_1(x_n + h, y_{1n} + l_1, y_{2n} + l_2), \\
p_2 &= hf_2(x_n + h, y_{1n} + l_1, y_{2n} + l_2).
\end{aligned}
$$

Example 2

Use RK-method 2nd order and 4th order to find the approximate solution of $y(0.1)$ and $z(0.1)$ as a solution of pair of equations

$$\begin{aligned}\frac{dy}{dx} &= x+z,\\ \frac{dz}{dx} &= y-x\end{aligned}$$

with the initial conditions

$$y(0)=1, \quad z(0)=-1.$$

Take step size $h=0.1$.

Multistep Methods

The general form of a linear m-step multistep method is

$$\begin{aligned}\frac{y_{n+1} - a_1 y_n - a_2 y_{n-1} - \cdots - a_m y_{n+1-m}}{h_i} \\ = b_0 f(x_{n+1}, y_{n+1}) + b_1 f(x_n, y_n) \\ + \cdots + b_m f(x_{n+1-m}, y_{n+1-m}).\end{aligned}$$

When $b_0 = 0$, the method is called explicit, otherwise, it is said to be implicit.

In multistep method, we require multiple starting values.

Adams-Bashforth Methods

We assume a uniform discretization in the x-domain, i.e., we define

$$x_i = a + ih,$$

where $h = \frac{b-a}{N}$, for some positive integer N.

Let the given differential equation is of the form

$$y'(x) = f(x, y).$$

We now integrate the given differential equation from $x = x_i$ to $x = x_{i+1}$.

This yields the equation

$$y(x_{i+1}) - y(x_i) = \int_{x_i}^{x_{i+1}} f(x, y(x))\, dx.$$

Next, we write

$$f(x, y(x)) = P_{m-1}(x) + R_{m-1}(x),$$

where

$$P_{m-1}(x) = \sum_{j=1}^{m} L_{m-1,j}(x) f(x_{i+1-j}, y(x_{i+1-j})),$$

is the Lagrange form of the polynomial of degree at most $m-1$ that interpolates f at the m points $x_i, x_{i-1}, x_{i-2}, \cdots, x_{i+1-m}$.

And

$$R_{m-1}(x) = \frac{f^{(m)}(\xi, y(\xi))}{m!} \Pi_{j=1}^{m} (x - x_{i+1-j})$$

is the corresponding remainder term.

Two-Step Adams Bashforth Method

Since $m = 2$, we write

$$
\begin{aligned}
f(x, y(x)) &= \frac{x - x_{i-1}}{x_i - x_{i-1}} f(x_i, y_i) \\
&+ \frac{x - x_i}{x_{i-1} - x_i} f(x_{i-1}, y_{i-1}) \\
&+ \frac{f''(\xi, y(\xi))}{2} (x - x_i)(x - x_{i-1}).
\end{aligned}
$$

Integrating the Lagrange polynomials yields

$$
\begin{aligned}
b_1 &= \int_{x_i}^{x_{i+1}} \frac{x - x_{i-1}}{x_i - x_{i-1}}\, dx = \frac{3h}{2}, \\
b_2 &= \int_{x_i}^{x_{i+1}} \frac{x - x_i}{x_{i-1} - x_i}\, dx = -\frac{h}{2}.
\end{aligned}
$$

Therefore, the two-step Adams-Bashforth method is

$$y_{n+1} = y_n + \frac{h}{2}\left[3f(x_n, y_n) - f(x_{n-1}, y_{n-1})\right].$$

Proceeding in a similar manner, the three step Adams-bashforth method is

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{12}[23f(x_n, y_n) - 16f(x_{n-1}, y_{n-1}) \\ &+ 5f(x_{n-2}, y_{n-2})]. \end{aligned}$$

The four-step Adams-Bashforth method is

$$
\begin{aligned}
y_{n+1} &= y_n + \frac{h}{24}[55f(x_n, y_n) - 59f(x_{n-1}, y_{n-1}) \\
&+ 37f(x_{n-2}, y_{n-2}) - 9f(x_{n-3}, y_{n-3})].
\end{aligned}
$$

Example 5

Use two-step Adams-Bashforth method to find the approximate solution of

$$\begin{aligned} \frac{dx}{dt} &= 1 + \frac{x}{t}, \\ x(1) &= 1, \end{aligned}$$

near $x(1.5)$. Take step size $h = 0.5$.

Solution

The two-step Adams-Bashforth formula is

$$x_{n+1} = x_n + \frac{h}{2}\left[3f(t_n, x_n) - f(t_{n-1}, x_{n-1})\right].$$

We note that for finding x_2, we require x_1 and x_0.

We calculate x_1 with the help of second order Taylor's method

$$x_1 = 2.125.$$

Now,

$$\begin{aligned} x_2 &= x_1 + \frac{h}{2}\left[3f(t_1, x_1) - f(t_0, x_0)\right], \\ &= 3.4375. \end{aligned}$$

Adams-Moulton Methods

The derivation of Adams-Moulton methods follows exactly the same procedure as the derivation of the Adams-Bashforth method with one exception.

In addition to interpolating f at $x_i, x_{i-1}, x_{i-2}, \cdots, x_{i+1-m}$, we also interpolate at x_{i+1}.

Hence, we write

$$f(x, y(x)) = P_m(x) + R_m(x),$$

where

$$P_m(x) = \sum_{j=0}^{m} L_{m,j}(x) f(x_{i+1-j}, y(x_{i+1-j})),$$

and

$$R_m(x) = \frac{f^{(m+1)}(\xi, y(\xi))}{(m+1)!} \Pi_{j=0}^{m} (x - x_{i+1-j}).$$

Since $P_m(x)$ contains a term involving $f(x_{i+1}, y(x_{i+1}))$, the resulting method will be implicit.

Furthermore, by using an additional point in the interpolating polynomial the degree of the remainder term is increased by one over the Adams-Bashforth case.

2-Step Adams-Moulton Method

Since $m = 2$, we write

$$f(x, y(x)) = \frac{(x - x_i)(x - x_{i-1})}{(x_{i+1} - x_i)(x_{i+1} - x_{i-1})} f(x_{i+1}, y_{i+1}) + \frac{(x - x_{i+1})(x - x_{i-1})}{(x_i - x_{i+1})(x_i - x_{i-1})} f(x_i, y_i) + \frac{(x - x_{i+1})(x - x_i)}{(x_{i-1} - x_{i+1})(x_{i-1} - x_i)} f(x_{i-1}, y_{i-1}) + \frac{f'''(\xi, y(\xi))}{6}(x - x_{i+1})(x - x_i)(x - x_{i-1}).$$

Integrating the Lagrange polynomials yields

$$
\begin{aligned}
b_0 &= \int_{x_i}^{x_{i+1}} \frac{(x - x_i)(x - x_{i-1})}{(x_{i+1} - x_i)(x_{i+1} - x_{i-1})}\, dt = \frac{5h}{12}, \\
b_1 &= \int_{x_i}^{x_{i+1}} \frac{(x - x_{i+1})(x - x_{i-1})}{(x_i - x_{i+1})(x_i - x_{i-1})}\, dt = \frac{2h}{3}, \\
b_2 &= \int_{x_i}^{x_{i+1}} \frac{(x - x_{i+1})(x - x_i)}{(x_{i-1} - x_{i+1})(x_{i-1} - x_i)}\, dt = -\frac{h}{12}.
\end{aligned}
$$

Therefore, the two-step Adams-Moulton method is

$$\frac{y_{i+1} - y_i}{h} = \frac{5}{12} f(x_{i+1}, y_{i+1})$$
$$+ \frac{2}{3} f(x_i, y_i) - \frac{1}{12} f(x_{i-1}, y_{i-1}).$$

To find the approximate solution, we required y_0 and y_1.

The initial condition give y_0 and y_1 can be obtained using any third order one-step method.

Similarly, we derive the three-step Adams-Moulton method is given by

$$y_{i+1} = y_i + \frac{h}{24}[9f(x_{i+1}, y_{i+1}) \\ +19f(x_i, y_i) - 5f(x_{i-1}, y_{i-1}) \\ +f(x_{i-2}, y_{i-2})].$$

The required starting values for the three step method should be obtained using a fourth-order one-step method.

Predictor-Corrector Method

The most popular of the predictor-corrector scheme is the Adams fourth-order predictor-corrector method.

This uses the four-step, fourth-order Adams-Bashforth method

$$\begin{aligned} y_{n+1}^{(0)} &= y_n + \frac{h}{24}[55f(x_n, y_n) - 59f(x_{n-1}, y_{n-1}) \\ &+ 37f(x_{n-2}, y_{n-2}) - 9f(x_{n-3}, y_{n-3})], \end{aligned}$$

as a predictor

followed by the three-step, fourth-order Adams-Moulton method

$$\begin{aligned} y_{n+1}^{(1)} &= y_n + \frac{h}{24}[9f(x_{n+1}, y_{n+1}^{(0)}) + 19f(x_n, y_n) \\ &- 5f(x_{n-1}, y_{n-1}) + f(x_{n-2}, y_{n-2})], \end{aligned}$$

as a corrector.

Note that the computation of $y_4^{(0)}$, we require the values $(x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3)$.

These values should be given with the given equation or should be computed using single step methods like, Euler's method, Taylor's method or Runge-Kutta methods.

To compute the value $y_4^{(1)}$, we require the values

$$(x_1, y_1), (x_2, y_2), (x_3, y_3) \text{ and } (x_4, y_4^{(0)}).$$

Modifier

Let $y(x_{n+1})$ denotes the true solution of y at x_{n+1}. Then

$$y(x_{n+1}) = y_{n+1}^{(0)} + \frac{251}{720}h^5 f^{(iv)}(\xi_1),\ x_{n-3} < \xi_1 < x_{n+1},$$

$$y(x_{n+1}) = y_{n+1}^{(1)} - \frac{19}{720}h^5 f^{(iv)}(\xi_2),\ x_{n-2} < \xi_2 < x_{n+1}.$$

On subtracting, we get

$$0 = y_{n+1}^{(0)} - y_{n+1}^{(1)} + \frac{270}{720}h^5 f^{(iv)}(\xi).$$

This implies that

$$y_{n+1}^{(1)} - y_{n+1}^{(0)} = \frac{270}{720} h^5 f^{(iv)}(\xi).$$

We rewrite as

$$\frac{h^5}{720} f^{(iv)}(\xi) = \frac{1}{270} (y_{n+1}^{(1)} - y_{n+1}^{(0)}).$$

Finally, we find the modifier as

$$y(x_{n+1}) - y_{n+1}^{(1)} = -\frac{19}{270}(y_{n+1}^{(1)} - y_{n+1}^{(0)}) = D_{n+1}.$$

Here, D_{n+1} is called as modifier.

Example

Find $y(1.4)$ by Adams Moulton 4th order predictor-corrector pair with modifier as a solution of

$$\frac{dy}{dx} = x^3 + xy,$$

$y(1) = 2,\ y(1.1) = 1.6,\ y(1.2) = 0.34$ and $y(1.3) = 0.594$ with spacing $h = 0.1$.