Chapter 7

Partial Differential Equations

7.2 Parabolic Differential Equations

7.2-2 Implicit Method

The unsteady state heat conduction in one dimension can be written as

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

The finite difference form of the heat equation is now given with the spatial second derivative evaluated from a combination of the derivatives at time steps (n) and (n+1)

$$\frac{1}{\alpha}\frac{T_i^{n+1} - T_i^n}{\Delta t} = f\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{(\Delta x)^2} + (1 - f)\frac{T_{i+1}^{n+1} - 2T_i^{n+1} + T_{i-1}^{n+1}}{(\Delta x)^2}$$

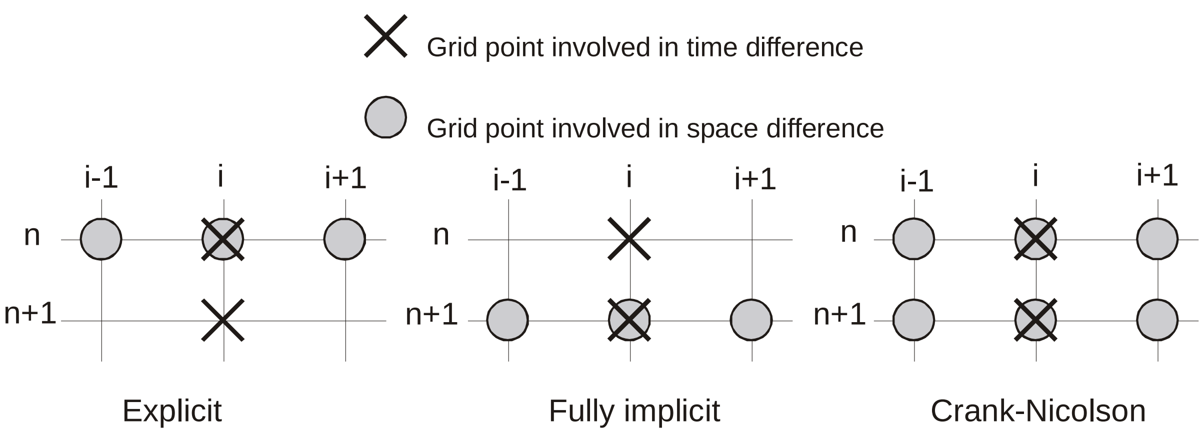

Figure 7.2-2 A computational diagram for explicit and implicit methods.

From the above formula, we will have an explicit method when f = 1 and a fully method when f = 0. In fact f can be any value between 0 and 1, however a common choice for f is 0.5. This is called the Crank-Nicolson method.

$$\frac{1}{\alpha}\frac{T_i^{n+1} - T_i^n}{\Delta t} = 0.5\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{(\Delta x)^2} + 0.5\frac{T_{i+1}^{n+1} - 2T_i^{n+1} + T_{i-1}^{n+1}}{(\Delta x)^2}$$

The equation can be arranged so that the temperatures at the present time step (n+1) are on the left hand side. Applying these equations to all the nodes, we will obtain a system with tridiagonal coefficient matrix.

$$- 0.5\frac{\alpha\Delta t}{(\Delta x)^2}\, T_{i-1}^{n+1} + \left(1 + 2\frac{\alpha\Delta t}{(\Delta x)^2}\right) T_i^{n+1} - 0.5\frac{\alpha\Delta t}{(\Delta x)^2}\, T_{i+1}^{n+1} = 0.5\frac{\alpha\Delta t}{(\Delta x)^2}\, T_{i-1}^{n} + \left(1 - 2\frac{\alpha\Delta t}{(\Delta x)^2}\right) T_i^{n} + \frac{\alpha\Delta t}{(\Delta x)^2}\, T_{i+1}^{n}$$

The expression $\frac{\alpha\Delta t}{(\Delta x)^2}$ is known as the diffusion number and will be denoted by d, the above equation becomes

$$- d T_{i-1}^{n+1} + 2(1 + d)\ T_i^{n+1} - d T_{i+1}^{n+1} = d T_{i-1}^{n} + 2(1 - d)\ T_i^{n} + d T_{i+1}^{n}$$

Example 7.2-2

Use the Crank-Nicolson method to solve the partial differential equation $\frac{\partial T}{\partial t} = 0.02\frac{\partial^2 T}{\partial x^2}$ with the following initial and boundary conditions:

Initial conditions: $T(x, 0) = 100x$ for $0 \le x \le 1$; $T(x, 0) = 100(2 - x)$ for $1 \le x \le 2$

Boundary conditions: $T(0, t) = 0$; $T(2, t) = 0$

Solution

Let $\Delta x = 0.2$, $\Delta t = 0.5$, we have

$$d = \frac{\alpha\Delta t}{(\Delta x)^2} = \frac{(0.02)(0.5)}{(0.2)^2} = 0.25 \le \frac{1}{2}$$

Substituting d into the equation

$$- d T_{i-1}^{n+1} + 2(1 + d)\ T_i^{n+1} - d T_{i+1}^{n+1} = d T_{i-1}^{n} + 2(1 - d)\ T_i^{n} + d T_{i+1}^{n}$$

we obtain

$$- 0.25 T_{i-1}^{n+1} + 2.5 T_i^{n+1} - 0.25 T_{i+1}^{n+1} = 0.25\ T_{i-1}^{n} + 1.5 T_i^{n} + 0.25 T_{i+1}^{n}$$

The initial conditions for the problem are given in the following table:

i	1	2	3	4	5	6
$t\backslash x$	0	0.2	0.4	0.6	0.8	1.0
0	0	20	40	60	80	100
0.5	0					

Due to the symmetry with respect to node $i = 6$, we have

$$-0.5T_5^{n+1} + 2.5T_6^{n+1} = 0.5T_5^n + 1.5T_6^n$$

The temperatures at the time t = 0.5 are the solutions of the following equations:

$i = 2$		$2.5T_2$	$-0.25\ T_3$	=	40
$i = 3$	$-0.25\ T_2$	$+ 2.5T_3$	$-0.25\ T_4$	=	80
$i = 4$	$-0.25\ T_3$	$+ 2.5T_4$	$-0.25\ T_5$	=	120
$i = 5$	$-0.25\ T_4$	$+ 2.5T_5$	$-0.25\ T_6$	=	160
$i = 6$	$-0.25\ T_5$	$+ 2.5T_6$		=	190

The results for the above set of equations are listed in the following table:

i	1	2	3	4	5	6
$t \backslash x$	0	0.2	0.4	0.6	0.8	1.0
0	0	20	40	60	80	100
0.5	0	20	39.99	59.92	79.18	91.84

Table 7.2-2 lists the MATLAB program and the results of the computation.

Table 7.2-2 An example of the Crank-Nicolson method ---------------

```
%
% Example 7.2-2, Crank-Nicolson method
%
Tn=[0 20 40 60 80 100];n=length(Tn);
n1=n-1;Tn1=Tn;
b=2.5*ones(1,n);
c=-0.25*ones(1,n);a=c;a(n)=2*a(n);
r=ones(1,n);
t=0;
fprintf('t = %8.1f, T = %8.2f %8.2f %8.2f %8.2f %8.2f %8.2f\n',t,Tn)
for k=1 : 10
  t=t+0.5;
for i=2:n1;
    r(i)=0.25*Tn(i-1)+1.5*Tn(i)+0.25*Tn(i+1);
end
r(6)=0.5*Tn(n1)+1.5*Tn(n);
beta=c;gam=c;y=c;
beta(2)=b(2);gam(2)=r(2)/beta(2);
for i=3:n
    beta(i)=b(i)-a(i)*c(i-1)/beta(i-1);
    gam(i)=(r(i)-a(i)*gam(i-1))/beta(i);
end
Tn1(n)=gam(n);
for j=1:n1-1
    Tn1(n-j)=gam(n-j)-c(n-j)*Tn1(n-j+1)/beta(n-j);
end
Tn=Tn1;
  fprintf('t = %8.1f, T = %8.2f %8.2f %8.2f %8.2f %8.2f %8.2f\n',t,Tn)
```

```
end

>> ex7d2d2
t =     0.0, T =     0.00    20.00    40.00    60.00    80.00   100.00
t =     0.5, T =     0.00    20.00    39.99    59.92    79.18    91.84
t =     1.0, T =     0.00    19.99    39.94    59.59    77.28    86.39
t =     1.5, T =     0.00    19.97    39.82    58.97    75.09    82.31
t =     2.0, T =     0.00    19.92    39.59    58.12    72.89    78.98
t =     2.5, T =     0.00    19.84    39.25    57.12    70.77    76.12
t =     3.0, T =     0.00    19.71    38.82    56.03    68.75    73.58
t =     3.5, T =     0.00    19.54    38.31    54.89    66.83    71.26
t =     4.0, T =     0.00    19.33    37.73    53.72    64.99    69.12
t =     4.5, T =     0.00    19.08    37.11    52.54    63.25    67.12
t =     5.0, T =     0.00    18.80    36.44    51.36    61.57    65.24
```

In the next example, we will show the formulation of the finite difference using Crank-Nicolson method for a parabolic equation with derivative boundary conditions.

Example 7.2-3 ---

Use the Crank-Nicolson method to solve the partial differential equation $\frac{\partial T}{\partial t} = 0.125\frac{\partial^2 T}{\partial x^2}$ with the following initial and boundary conditions:

Initial conditions: $T(x, 0) = 1000$, $0 \le x \le 4$

Boundary conditions: $\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0.36(T - 70)$; $\left.\frac{\partial T}{\partial x}\right|_{x=4} = 0$

Solution

Using the Crank-Nicolson method the heat equation becomes

$$\frac{1}{\alpha}\frac{T_i^{n+1} - T_i^n}{\Delta t} = 0.5\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{(\Delta x)^2} + 0.5\frac{T_{i+1}^{n+1} - 2T_i^{n+1} + T_{i-1}^{n+1}}{(\Delta x)^2}$$

Let $d = \frac{\alpha\Delta t}{(\Delta x)^2}$, the above equation can be rearranged to

$$-dT_{i-1}^{n+1} + 2(1 + d)T_i^{n+1} - dT_{i+1}^{n+1} = dT_{i-1}^n + 2(1 - d)\ T_i^n + dT_{i+1}^n$$

Let $\Delta x = 1$, $\Delta t = 1$, we have

$$d = \frac{\alpha\Delta t}{(\Delta x)^2} = 0.125 \le \frac{1}{2}$$

Substituting d into the above equation we obtain

$$-0.125T_{i-1}^{n+1} + 2.25T_i^{n+1} - 0.125T_{i+1}^{n+1} = 0.125T_{i-1}^n + 1.75T_i^n + 0.125T_{i+1}^n$$

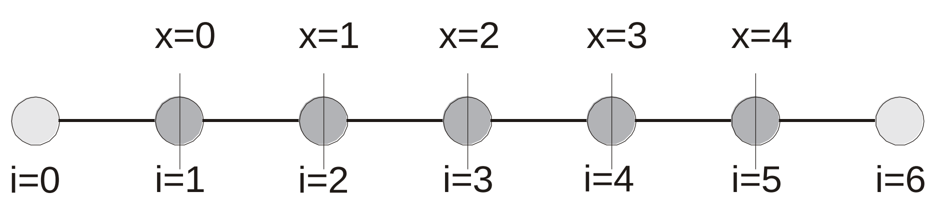

Figure 7.2-3 The nodes at any given time step.

There are five unknown temperatures at any given time as shown in Figure 7.2-3. The nodes corresponding to i = 0 and i = 5 are required to accommodate the derivative boundary conditions at the end points.

For node (i = 1)

$$- 0.125\,T_0^{n+1} + 2.25\,T_1^{n+1} - 0.125\,T_2^{n+1} = 0.125\,T_0^{n} + 1.75\,T_1^{n} + 0.125\,T_2^{n}$$

T_0^{n+1} can be solved in terms of T_1^{n+1} and T_2^{n+1} using the boundary condition

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0.36(T - 70) \Rightarrow \frac{1}{2\Delta x}(T_2^{n+1} - T_0^{n+1}) = 0.36(T_1^{n+1} - 70)$$

$$T_0^{n+1} = T_2^{n+1} - 0.72\,T_1^{n+1} + 50.4$$

Similarly, T_0^{n} can be solved in terms of T_1^{n} and T_2^{n} using the same boundary condition

$$T_0^{n} = T_2^{n} - 0.72\,T_1^{n} + 50.4$$

The equation at node (1) is then simplified to

$$2.34\,T_1^{n+1} - 0.25\,T_2^{n+1} = 1.66\,T_1^{n} + 0.25\,T_2^{n} + 12.6$$

For $2 \le i \le 4$

$$- 0.125\,T_{i-1}^{n+1} + 2.25\,T_i^{n+1} - 0.125\,T_{i+1}^{n+1} = 0.125\,T_{i-1}^{n} + 1.75\,T_i^{n} + 0.125\,T_{i+1}^{n}$$

At node (5) $\left.\frac{\partial T}{\partial x}\right|_{x=4} = 0$, therefore $T_4 = T_6$

$$- 0.25\,T_4^{n+1} + 2.25\,T_5^{n+1} = 0.25\,T_4^{n} + 1.75\,T_5^{n}$$

Table 7.2-3 lists the MATLAB program and the results of the computation.

Table 7.2-3 An example of the Crank-Nicolson method ---------------

```
%
% Example 7.2-3, Crank-Nicolson method with derivative boundary conditions
%
Tn=1000*ones(1,5);n=length(Tn);
n1=n-1;Tn1=Tn;
b=2.25*ones(1,n);b(1)=2.34;
c=-0.125*ones(1,n);a=c;c(1)=2*c(1);a(n)=2*a(n);
r=ones(1,n);
t=0;
fprintf('t = %8.1f, T = %8.2f %8.2f %8.2f %8.2f %8.2f\n',t,Tn)
for k=1 : 10
  t=t+1;
  r(1)=1.66*Tn(1)+.25*Tn(2)+12.6;
for i=2:n1;
    r(i)=0.125*Tn(i-1)+1.75*Tn(i)+0.125*Tn(i+1);
```

```
end
r(5)=0.25*Tn(n1)+1.75*Tn(n);
beta=c;gam=c;y=c;
beta(1)=b(1);gam(1)=r(1)/beta(1);
for i=2:n
   beta(i)=b(i)-a(i)*c(i-1)/beta(i-1);
   gam(i)=(r(i)-a(i)*gam(i-1))/beta(i);
end
Tn1(n)=gam(n);
for j=1:n1
   Tn1(n-j)=gam(n-j)-c(n-j)*Tn1(n-j+1)/beta(n-j);
end
Tn=Tn1;
 fprintf('t = %8.1f, T = %8.2f %8.2f %8.2f %8.2f %8.2f\n',t,Tn)
end

>> ex7d2d3
t =      0.0, T =  1000.00  1000.00  1000.00  1000.00  1000.00
t =      1.0, T =   928.03   995.99   999.78   999.99  1000.00
t =      2.0, T =   875.47   985.89   998.81   999.91   999.99
t =      3.0, T =   835.70   972.73   996.76   999.68   999.95
t =      4.0, T =   804.53   958.27   993.58   999.20   999.83
t =      5.0, T =   779.30   943.48   989.42   998.40   999.60
t =      6.0, T =   758.26   928.89   984.44   997.24   999.21
t =      7.0, T =   740.27   914.79   978.82   995.69   998.60
t =      8.0, T =   724.56   901.30   972.72   993.75   997.74
t =      9.0, T =   710.61   888.47   966.28   991.44   996.59
t =     10.0, T =   698.03   876.28   959.60   988.76   995.15
```
