The Linear Shooting Method - (8.4)

Consider the boundary value problems (BVPs) for the second order differential equation of the form

$$(*) \qquad y'' = f(x,y,y'), \quad a \le x \le b,\ y(a) = \alpha \text{ and } y(b) = \beta.$$

Under what conditions a boundary value problem has a solution or has a unique solution.

1. **Existence and Uniqueness**:

Suppose that f is continuous on the set

$$D = \left\{(x,y,y');\ a \le x \le b,\ \ -\infty < y < \infty,\ \ -\infty < y' < \infty\right\}$$

and the partial derivatives f_y and $f_{y'}$ are also continuous on D. If

(i) $f_y(x,y,y') > 0$, for all $\left(x,y,\ y'\right)$ in D, and

(ii) there exists a constant M such that

$$\left|f_{y'}(x,y,y')\right| \le M \text{ for all } (x,y,y') \text{ in } D,$$

then the boundary value problem (*) has a **unique solution**.

Example *Consider the following boundary value problem:*

$$y'' + e^{-xy} + \sin(y') = 0,\ 1 \le x \le 2,\ y(1) = y(2) = 0$$

Determine if the boundary value problem has a unique solution.

Rewrite $y'' = -e^{-xy} - \sin(y')$ so $f(x,y,y') = -e^{-xy} - \sin(y')$

Check conditions:

$f(x,y,y') = -e^{-xy} - \sin(y')$, $f_y(x,y,y') = xe^{-xy}$, and $f_{y'}(x,y,y') = -\cos(y')$ are continuous on

$$D = \left\{(x,y,y');\ \ 1 \le x \le 2,\ \ -\infty < y < \infty,\ \ -\infty < y' < \infty\right\}.$$

(i) $f_y(x,y,y') = xe^{-xy} > 0$ on D.

(ii) $\left|f_{y'}(t,y,y')\right| = |-\cos(y')| \le 1 = M$.

So, the boundary value problem has a unique solution in D.

Example *Consider the linear boundary value problem:*

$$y'' = p(x)y' + q(x)y + r(x),\ a \le x \le b, \quad y(a) = \alpha,\ y(b) = \beta$$

Under what condition(s) a linear BVP has a unique solution?

$f(x,y,y') = p(x)y' + q(x)y + r(x)$, $f_y(x,y,y') = q(x)$, $f_{y'}(x,y,y') = p(x)$ are continuous on D if $p(x)$, $q(x)$ and $r(x)$ are continuous for $a \le x \le b$.

a. $f_y(x,y,y') = q(x) > 0$ for $a \le x \le b$.

b. Since $f_{y'}$ is continuous on $\left[a,\ b\right]$, $f_{y'}$ is bounded.

So, if $p(x)$, $q(x)$ and $r(x)$ are continuous for $a \le x \le b$, and $q(x) > 0$ for $a \le x \le b$, then the boundary value problem has a unique solution.

2. **The Linear Shooting Method**:

Consider the linear boundary value problems of the form:

$$y'' = p(x)y' + q(x)y + r(x), \quad a \le x \le b,\ y(a) = \alpha,\ y(b) = \beta$$

where $p(x)$, $q(x)$ and $r(x)$ are continuous and $q(x) > 0$ for $a \le x \le b$. Consider the solutions of the following two initial-value problems:

$$(**)\quad \begin{array}{ll} \text{(i)} & y'' = p(x)y' + q(x)y + r(x), \quad a \le x \le b, \quad y(a) = \alpha, \ y'(a) = 0 \\ \text{(ii)} & y'' = p(x)y' + q(x)y, \quad a \le x \le b, \ y(a) = 0, \ y'(a) = 1 \end{array}$$

say, $y_1(x)$ and $y_2(x)$. Let $y(x)$ be the following linear combination of $y_1(x)$ and $y_2(x)$:

$$y(x) = y_1(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2(x)$$

Then $y(x)$ is the solution of the boundary value problem. Check:

$$y''(x) = y_1''(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2''(x) = p(x)y_1'(x) + q(x)y_1(x) + r(x) + \frac{\beta - y_1(b)}{y_2(b)} (p(x)y_2'(x) + q(x)y_2(x))$$

$$= p(x)\left(y_1'(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2'(x)\right) + q(x)\left(y_1(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2(x)\right) + r(x)$$

So, $y(x)$ is a solution of $y'' = p(x)y' + q(x)y + r(x)$. Check the boundary conditions:

$$y(a) = y_1(a) + \frac{\beta - y_1(b)}{y_2(b)} y_2(a) = \alpha + \frac{\beta - y_1(b)}{y_2(b)} (0) = \alpha$$

$$y(b) = y_1(b) + \frac{\beta - y_1(b)}{y_2(b)} y_2(b) = \beta.$$

This suggests that a boundary value problem can be solved by solving two (independent) initial-value problems in (**).

Review: Solve a second-order initial-value problem:

$$y'' = f(x,y,y'), \qquad y(a) = \alpha_0, \ y'(a) = \alpha_1.$$

Let $u_1 = y_1$, and $u_2 = y_1'$. Then above second-order differential equation for y becomes the following system of two first-order differential equation in u_1 and u_2:

$$\begin{cases} u_1' = u_2 \\ u_2' = f_1(x, u_1, u_2) \end{cases}, \quad a \le x \le b, \ u_1(a) = \alpha_0, \ u_2(a) = \alpha_1.$$

Example *Rewrite the differential equation $y'' - 2y' + 4y = te^{2t}$ as a system of 2 1st-order differential equations.*

Set

$u_1 = y$	$\Rightarrow$	
$u_2 = y' = u_1'$	$\Rightarrow$	$u_1' = y'$
	$\Rightarrow$	$u_2' = y'' = 2y' - 4y + te^{2t} = -4u_1 + 2u_2 + te^{2t}$

The system is:

$$\begin{cases} u_1' = u_2 \\ u_2' = -4u_1 + 2u_2 + te^{2t} \end{cases} \Rightarrow \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 \\ te^{2t} \end{bmatrix}$$

Example *Rewrite the initial-value problem for the system of* **2** *second-order differential equations*

$$\begin{cases} y_1'' + y_1' - 2y_2' + 2y_1 - y_2 = t \\ y_2'' - 3y_1' + y_1 + 4y_2 = 2\sin(t) \end{cases}, \; y_1(0) = 1,\; y_2(0) = -1,\; y_1'(0) = 2,\; y_2'(0) = 3$$

as an initial-value problem for a system of **4** *first-order differential equations.*

Set

$u_1 = y_1$		
$u_2 = y_1' = u_1'$	$\Rightarrow$	$u_1' = u_2$
	$\Rightarrow$	$u_2' = y_1'' = -2y_1 + y_2 - y_1' + 2y_2' + t = -2u_1 + u_3 - u_2 + 2u_4 + t$
$u_3 = y_2$		
$u_4 = y_2' = u_3'$	$\Rightarrow$	$u_3' = u_4$
	$\Rightarrow$	$u_4' = y_2'' = -y_1 - 4y_2 + 3y_1' + 2\sin(t) = -u_1 - 4u_3 + 3u_2 + 2\sin(t)$

The system of 4 1st-order linear differential equations is:

$$\begin{cases} u_1' = u_2 \\ u_2' = -2u_1 + u_3 - u_2 + 2u_4 + t \\ u_3' = u_4 \\ u_4' = -u_1 - 4u_3 + 3u_2 + 2\sin(t) \end{cases}, \; u_1(0) = 1,\; u_2(0) = 2,\; u_3(0) = -1,\; u_4(0) = 3;$$

or in matrix-vector notation:

$$\begin{bmatrix} u_1' \\ u_2' \\ u_3' \\ u_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & -1 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ -1 & 3 & -4 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ 0 \\ 2\sin(t) \end{bmatrix}, \quad \begin{bmatrix} u_1(0) \\ u_2(0) \\ u_3(0) \\ u_4(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$$

Example *Solve the boundary value problem:*

$$y'' = -\frac{4}{x}y' - \frac{2}{x^2}y + \frac{2}{x^2}\ln x, \quad 1 \le x \le 2,\; y(1) = \frac{1}{2},\; y(2) = \ln 2.$$

Exact solution is:

$$y(x) = -\frac{1}{x^2}\left(2 - 4x + \frac{3}{2}x^2 - x^2\ln x\right).$$

Note that it is a linear boundary value problem where $p(x) = -\frac{4}{x},\; q(x) = -\frac{2}{x^2},\; r(x) = \frac{2}{x^2}\ln x$ continuous on $[1,2]$.

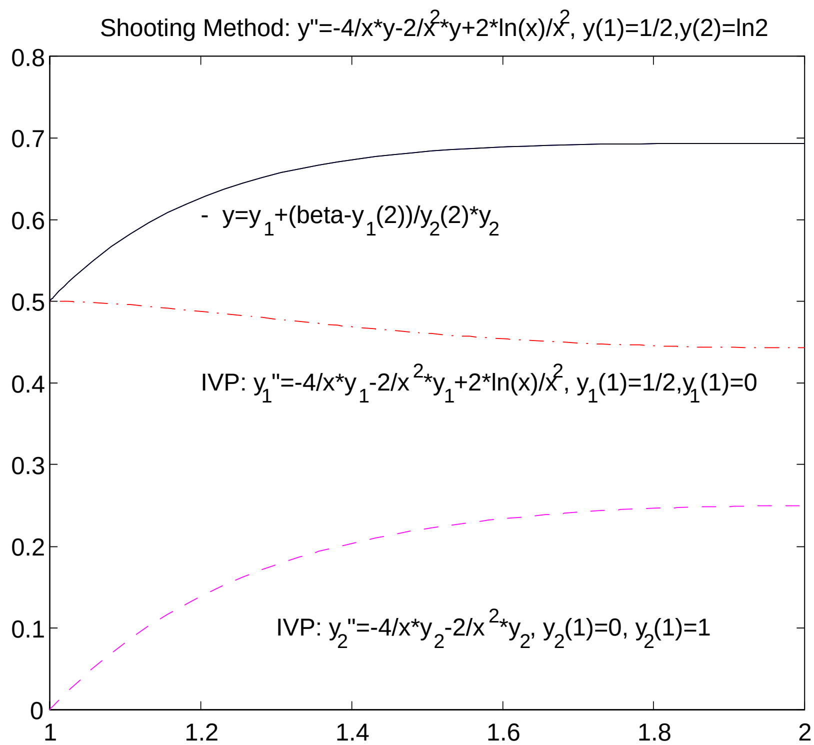
Since $q(x) \ngeq 0$, we cannot say if this boundary value problem has a unique solution. Now we solve the following two initial-value problems:

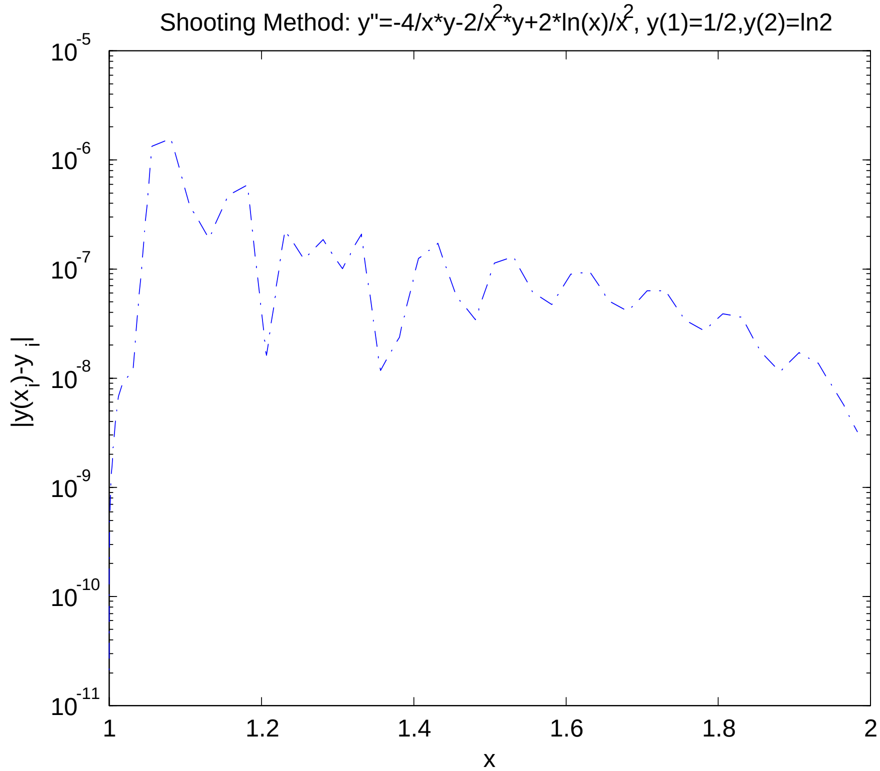
$$\text{(i)} \quad y_1'' = -\frac{4}{x}y_1' - \frac{2}{x^2}y_1 + \frac{2}{x^2}\ln x, \quad 1 \le x \le 2, \quad y_1(1) = \frac{1}{2},\; y_1'(1) = 0$$

$$\text{(ii)} \quad y_2'' = -\frac{4}{x}y_2' - \frac{2}{x^2}y_2, \quad 1 \le x \le 2,\; y_2(1) = 0,\; y_2'(1) = 1$$

Set up the initial-value problem for a system of 4 1st-order differential equations:

$$\begin{cases} u_1' = u_2 \\ u_2' = -\dfrac{4}{x}u_2 - \dfrac{2}{x^2}u_1 + \dfrac{2}{x^2}\ln x \\ u_3' = u_4 \\ u_4' = -\dfrac{4}{x}u_4 - \dfrac{2}{x^2}u_3 \end{cases}, \; u_1(1) = \frac{1}{2},\; u_2(1) = 0,\; u_3(1) = 0,\; u_4(1) = 1.$$





Shooting Method: y''=-4/x*y-2/x^2*y+2*ln(x)/x^2, y(1)=1/2,y(2)=ln2
10^-5
10^-6
10^-7
10^-8
10^-9
10^-10
10^-11
|y(x_i)-y_i|
1
1.2
1.4
1.6
1.8
2
x

MatLab program for this example:

```
clf
alpha=1/2;
beta=log(2);
a=1;
b=2;
[xv,yv]=ode45('funsysa',[a b],[alpha;0;0;1]);
plot(xv,yv(:,1),'r-.',xv,yv(:,3),'m–')
hold
n=length(yv(:,1));
y1n=yv(n,1);
y2n=yv(n,3);
yvsol=yv(:,1)+(beta-y1n)/y2n*yv(:,3);
truesol=-1./(xv.^2).*(2-4*xv+3/2*xv.^2-xv.^2.*log(xv));
plot(xv,yvsol,'b-',xv,truesol,'k-')
title('Shooting Method: y"=-4/x*y-2/x^2*y+2*ln(x)/x^2, y(1)=1/2,y(2)=ln2')
text(1.2,0.4,'IVP: y_1"=-4/x*y_1-2/x^2*y_1+2*ln(x)/x^2, y_1(1)=1/2,y_1(1)=0')
text(1.3,0.1,'IVP: y_2"=-4/x*y_2-2/x^2*y_2, y_2(1)=0, y_2(1)=1')
text(1.2,0.6,'- y=y_1+(beta-y_1(2))/y_2(2)*y_2')
axis([1 2 0 0.8])
hold off
figure(2)
semilogy(xv,abs(yvsol-truesol),'b-.')
title('Shooting Method: y"=-4/x*y-2/x^2*y+2*ln(x)/x^2, y(1)=1/2,y(2)=ln2')
ylabel('|y(x_i)-y_i| ')
xlabel('x')
```

MatLab function: funsysa.m

```
function yv=funsysa(t,y);
yv(1,1)=y(2,1);
yv(2,1)=-4/t*y(2,1)-2/(t^2)*y(1,1)+2*log(t)/(t^2);
yv(3,1)=y(4,1);
yv(4,1)=-4/t*y(4,1)-2/(t^2)*y(3,1);
```

Notes: **ode45.m** is a MatLab building-in function which solves initial-value problems for systems of n first-order differential equations:

$$\begin{cases} u_1' = f_1(x,u_1,\ldots,u_n) \\ \qquad\vdots \\ u_n' = f_n(x,u_1,\ldots,u_n) \end{cases}, \quad a \le x \le b, \;\; u_1(a) = \alpha_1, \;\ldots,\; u_n(a) = \alpha_n.$$

It is called by:

[**xv**,**uv**]=**ode45**('**funsysa**',[**a b**],[$\alpha_1,\ldots,\alpha_n$]); where **funsysa.m** is a user-provided Matlab program that evaluates functions of the system at x. The outputs are the vector $xv = \begin{bmatrix} x_0 = a \\ x_1 \\ \vdots \\ x_{N-1} \\ x_N = b \end{bmatrix}$ and

$uv = \begin{bmatrix} u_{10} & \cdots & u_{n0} \\ u_{11} & \cdots & u_{n1} \\ \vdots & \vdots & \vdots \\ u_{1N} & \cdots & u_{nN} \end{bmatrix}$. That is, for $a \le x \le b$,

$$y_1(x) \approx \{u_{10},\ u_{11},\ \ldots,\ u_{1N}\},\ \ \ldots,\ \ y_n(x) \approx \{\, u_{n0},\ u_{n1},\ \ldots,\ u_{nN}\,\}.$$