Physics 105a: Mathematical and Computational Physics

Daniel Green

Department of Physics, University of California, San Diego, La Jolla, CA 92093, USA

A combined analytic and mathematically-based numerical approach to the solution of common applied mathematics problems in physics and engineering. Topics: Fourier series and integrals, special functions, initial and boundary value problems, Green's functions; heat, Laplace and wave equations.

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1 Motivation

Courses of this type are often described as "math methods" courses. This description seems uninspired and raises two obvious questions: (1) why is the physics department teaching a math course (physics students can surely handle a course in the math department) and (2) is this just a course in memorizing a list of specific mathematic results?

Mathematical and computational physics are instead important parts of a physics department for good reason and it is my hope that this course will give you some flavor of it (while teaching you want you need to know for your future endeavors). Although we teach specific fields and topics, physics, and really all of science, is a united endeavor. One guise of this unity is that we see essentially the same phenomena appear in seemingly completely different areas. Relatedly, there are numerous examples of insights, originally invented to solve one specific problem, that have had profound impact far beyond their original application. One reason for the far reaching impact of these ideas is that the mathematical description of these different phenomena are either the same or closely related. As a result, the behavior ends up looking the same. This unity of physical phenomena is part of what makes scientific knowledge so valuable.

Mathematical physics can be thought of as the unified description of all of these different physical phenomena. It is not divorced from physical intuition and the physical world but is enhanced by the fact that many different pieces of physical intuition can be used to understand the same problems. For example, when you think about friction, you can think either about a block sliding on an inclined plane or current running through a resistor. Relatedly, we know that (LCR) AC circuits behave like a ball on a spring. While you might have no intuition for AC circuits, you probably have some feeling for what to expect of the ball. Even more dramatically, we will see how the mathematical approach to understanding this problem (circuits, oscillators) will bring tools to bear on problem like data analysis and image processing and far beyond what one can even see in the physics department. I suspect that most of you will see some / all of these topics reappear throughout our careers, no matter where the future takes you.

A striking modern example is the interplay between research into artificial intelligence (AI) and statistical physics. AI has profoundly influenced physics research by helping automate and/or speed up a number of complex tasks. In return, many techniques in AI correspond to finding the lowest energy state in a very complex potential, which is essentially the description of a glass. In fact, in some specific cases, AI techniques have been shown to be mathematically equivalent to specific problems in statistical physics.

1.1 Practical Implications of Mathematical Physics

While the conceptual mysteries of the world/universe are what excite many physicists/scientists about their day-to-day work, the reality is that we only get to enjoy learning more about these big picture questions if we can solve the technical problems that stand in the way of progress. Perhaps the simplest example are the pictures of space, (e.g. like from the Hubble space telescope) that capture the public's imagination of science. In most cases, the raw data is filled with all kinds of noise and unwanted signals. To produce these beautiful images, we often have to filter out the noise and/or separate the signal components we want from those we don't. How do you

do this? Without having a powerful mathematical toolkit at your disposal, the beautiful images would never emerge from your data. Some version of this is true in every aspect of science: at the heart of every beautiful result is a method to solve a difficult technical problem.

Now, I can also see how one could look at this type of example and say "but I'm not interested in image processing, so should I care how this is done?". One could surely ask the same question of every problem, but this one illustrates why this is the wrong attitude. The same ideas behind how we filter data of this type show up in all kinds of different situations: they also appear in everything from the origin of galaxies to fractals and chaos. Good ideas get recycled over and over again.

This course is not going to teach you every tool you will ever need (no course could ever do that). However, the goal is to teach you how the same mathematical tools appear in different guises and, hopefully, will help you recognize what tricks work for what kinds of problems. At some point you may see a problem that has a similar "feel" to it and you might think that a certain approach would help. This is how problem solving in the real world happens.

1.2 A Current Example

One last piece of inspiration comes from the physics of phase transitions. There are lots of kinds of phase transitions that you know well, like the transition that occurs when you boil water. Perhaps some of you even know that there is a critical point in pressure, beyond which the liquid and gas phases actually are the same (something like boiling doesn't happen as you heat up the water). If you zoom-in on the physics right at the critical point itself, you get some unusual behavior because it is a special kind of phase transition known as a continuous or second order phase transition.

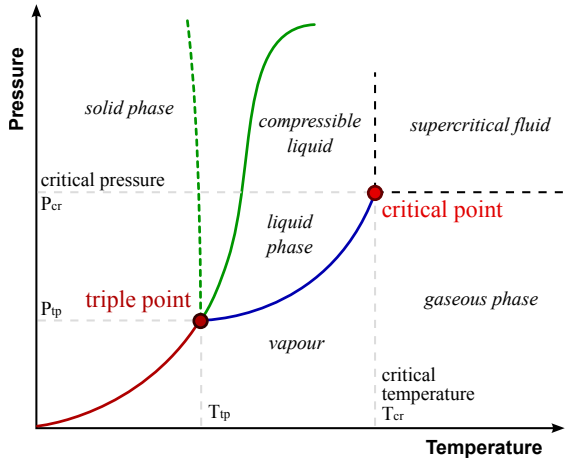

Figure 1: Sketch of the phase diagram of a typical substance.

So far, it sounds like we are just taking about some obscure facts about fluids, perhaps only of interest to phase diagram aficionados. That was perhaps even true until Ken Wilson came along, and showed us that the physics of these points is actually exactly the same as all kinds of other phenomena that one would never imagine are the same. Evolution of populations of

animals, the behavior of sandpiles, the central limit theorem (random numbers), the properties of fractals, and on and on and on: these all exhibit the phase transitions that are mathematically similar, or even equivalent, to the critical point of water.

What happens at these points is that the behavior of the systems becomes independent of scale. What this means is that if I make the system twice the size, it looks the same in a very precise mathematical sense. A fractal is one example you might know: it looks the same when you zoom in or out, even though its appearance is very complex. Once you have a system that obeys some kind of scaling behavior, the microscopic details of the system become irrelevant; whether the individual elements are atoms, cells, or people the end result is still the same. To take an extreme example from my own field, cosmology, the behavior of groups of billions of galaxies interacting through gravity on the scale of the entire universe can be understood from the same equations that describe the flow of water. I recommend the book Scale by Geoffery West for examples ranging from the lifespan of animals to city planning.

The importance of these connections between different statistical systems has been very well illustrated by the response to COVID-19. Since the beginning of the pandemic, researchers from all branches of science dropped what they were doing to try to understand the evolution of the disease and recommend strategies to mitigate it. The large scale behavior of a disease in a network of people is not fundamentally different than the collective behaviors witnesses in physical, biological or economic systems. From the right mathematical perspective, knowledge of those systems has lead to valuable insights that has influenced policy decisions around the world[1].

1.3 Waves: the unifying theme of this course

If you understand one thing about physics really well, it should be waves. They appear in one way or another in almost every physical system from light and sound, to quantum mechanics and the beginning of the universe. Waves are so prevalent, that much of how we think about the world is describe in a language built for waves. This perspective will help us organize the topics in this course:

- **The Fourier Transform:** This is just the natural language for discussing waves, break wave up into a super-position of waves with specific frequencies and wavelengths (breaking light up into colors). The fourier transform is way to apply this to any problem. At the level of problem solving, it turns the wave equation from an partial differential equation (PDE) to an algebraic equation. Once you see the power of the fourier transform, you realize that you can talk about any function in terms of its frequency and wavelength.

- **Complex Analysis:** Waves are easy to understand in fourier space, but we live in position space. Going from one to the over requires integrals over complex valued functions.

- **Green's Functions:** We have a source of light / sound in some complicated medium. The Green's function describes what happens to the waves after we create them.

[1] If you are skeptical, you can take it up with UCSD Physics Professor Nigel Goldenfeld, who has made profound contributions to all of these topics, including modeling COVID-19 and how universities should respond to it.

- **Partial Differential Equations:** The world evolves in more than one-dimension. When we think of confined waves or bound atoms, the boundary conditions might not make fourier space very useful. What other tools do we have at our disposal? Like to fourier transform, we will find more general ways to convert ODEs to algebraic equations.
- **Spherical Harmonics and Special Functions:** Waves are sometime confined in spherically symmetry potentials or the surface of a planet. For such cases, spherical harmonics are the natural generalization of the fourier transform.

As we go through the course, I will try to emphasize the physical connection between the tools we are developing and why it is a good way to talk about physical systems. I am trying to give you a map of where we are going in the hopes that I won't lose you along the way.

2 Introduction and Review

Dubin Chapter 1, Sections 1.6, 1.3

Every idea that we develop in this course is a generalization of something you have already seen for simple harmonic oscillators. We will review the simple harmonic oscilator with an eye towards these features so that we can understand how and why they reappear in more complex examples.

2.1 The undisturbed harmonic oscillator

The harmonic oscillator is by far the most ubiquitous phenomena in physics. Nearly every physical system contains some component that behaves like a harmonic oscillator. As such, we study it to death because every piece of intuition you can develop will serve you will in some situation or another.

The simplest incarnation of the harmonic oscillator is a one-dimensional frictionless spring. We have an object of mass m at position x and apply a linear restoring force $F = -\kappa x$ so that our equation is

$$m\ddot{x}(t) = -\kappa x(t) \rightarrow \ddot{x}(t) + \omega_0^2 x(t) = 0 \tag{2.1}$$

where $\omega_0^2 \equiv \kappa/m$ and $\dot{x}(t) = \frac{d}{dt}x(t)$, $\ddot{x}(t) = \frac{d^2}{dt^2}x(t)$, etc. We can easily check that this is solved by $x(t) = a\sin\omega_0 t + b\cos\omega_0 t$.

Of course, the world is more complicated and we need to include friction, as follows:

$$\ddot{x}(t) + \gamma\dot{x}(t) + \omega_0^2 x(t) = 0 \tag{2.2}$$

Note that γ and ω_0 have the same units of $[\text{time}]^{-1}$. To find the solution to this equation, we start by making the standard guess that there is a the solution of the from $x(t) = Ae^{\rho t}$ where A and ρ are parameters we hope are determined using the equation of motion. Plugging this *ansatz* into our equation we have

$$[\rho^2 + \gamma\rho + \omega_0^2]x(t) = 0 \rightarrow \rho = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2} \tag{2.3}$$

When $\omega_0 > \gamma/2$ the second term is imaginary, so it is useful to write

$$\rho = \pm i\omega - \gamma/2 \qquad \text{where} \qquad \omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}\ . \tag{2.4}$$

This all looks good, but $x(t)$ is a *real* physical quantity but in the oscillating phase I have

$$x(t) = Ae^{i\omega t}e^{-\gamma t/2} \tag{2.5}$$

which is complex. Clearly I did something wrong, but to get there, let's remember some useful facts about complex numbers.

[If instead $0 < \omega_0 < \gamma/2$, the we find two exponentially decaying solutions but the slower decay is given by

$$x(t) \approx Ae^{-\gamma/2+\sqrt{\frac{\gamma^2}{4}-\omega_0^2}} \rightarrow 0\ . \tag{2.6}$$

This is usually called the over-damped solution because the damping is too strong for it to oscillate.]

Aside: Complex Numbers: *I am going to assume that you have all seen complex numbers, at least in their basic form. We will learn more about complex numbers in this course, but let me remind you of some basic and very important facts about complex numbers:*

1. *A complex number $z = x + iy$ has a real part $\mathrm{Re}z = x$ and imaginary part $\mathrm{Im}z = y$ that are both real numbers and $i^2 = -1$.*

2. *The complex conjugate $z^* = x - iy$ comes from switching $i \to -i$.*

3. *The "absolute value" is a very useful thing to keep track of $\sqrt{zz^*}$. $zz^* = x^2 + y^2$ is always a positive real number.*

4. *You can get the real part just by adding z and its complex conjugate $x = (z + z^*)/2$ or the imaginary part by taking the difference $y = -i(z - z^*)/2$ (remember $i(-i) = 1$).*

5. *Complex numbers are extremely useful because of the Fundamental Theorem of Algebra: any non-constant polynomial equation of one variable with complex coefficients has a complex solution! Let me illustrate. Consider the equation:*

$$a^2 + b^2 = 0 \tag{2.7}$$

   If a and b are real numbers, then there is only one solution $a = b = 0$. I started with one equation and two unknown and I somehow determined both a and b. Alternatively, we could write $a^2 + b^2 = -1$ and find no solution at all! However, if a and b are complex numbers than $a = \pm ib$ are solutions and so I have determined one unknown from one equation. I.e. one equation gives me the relation between the two unknowns, a and b, but I still need to determine either a or b to fully solve the problem. What is amazing about complex numbers is that this is always true: every (polynomial) equation has a complex solution. Notice that this is the first kind of number you encountered with this property (polynomials with integer/rational/real coefficients have solutions that are not integer/rational/real).

6. *The next most important fact is Euler's formula: $e^{ix} = \cos(x) + i\sin(x)$. It is easiest to get this just by writing out the Taylor series. It is a remarkable mathematical fact that connects the worlds of algebra and geometry. It is also extremely useful for a lot of reasons, as we will see in this course. The simplest demonstration is that you can very easily show all kinds of trig identities just using what you know about exponentials: $e^a e^b = e^{a+b}$ and $(e^a)^b = e^{ab}$. For example:*

   - $\mathrm{Re}[(e^{ix})^2] = \cos 2x = \mathrm{Re}[(\cos x + i\sin x)^2] = \cos^2 x - \sin^2 x$
   - $e^{ix}e^{-ix} = e^0 = 1 = (\cos x + i\sin x)(\cos x - i\sin x) = \cos x^2 + \sin^2 x$
   - $e^{ix}e^{iy} = \cos(x+y) + i\sin(x+y) = (\cos x + i\sin x)(\cos y + i\sin y) = (\cos x\cos y - \sin x\sin y) + i(\cos x\sin y + \sin x\cos y)$
   - $\partial_x e^{ix} = ie^{ix} = -\sin(x) + i\cos(x) \to \partial_x \cos x = -\sin x \qquad \partial_x \sin x = \cos x$.

7. *You can also write a complex number in polar form $z = \rho e^{i\theta}$. This determines the x and y position in a complex plane by taking a circle of radius ρ and a taking the point at an angle θ around the circle counter-clockwise, starting at $z = +\rho$. This is just a different way of thinking about Euler's formula.*

Complex numbers are going to show up a lot in this course. Sometimes it will just be an easier way to find a solution to an equation (like the harmonic oscillator). At other times, the physical quantities will actually be complex numbers (like in quantum mechanics or the fourier transform).

Now we know how to fix our problem. We want $x(t)$ to be real, but we picked $x(t) = Ae^{\rho t}$ and found out that ρ was complex. We can fix that problem by letting A be a complex number and remembering we can add solutions so that $x(t) = Ae^{\rho t} + (Ae^{\rho t})^*$ so that $x(t)$ is real, even though we are using complex solutions. This works for two reasons

1. Because the equation has real coefficients, we can take the complex conjugate of the entire equation as follows: suppose I have a complex solution $f(t)$ then when I take the complex conjugate of the equation (complex conjugate both sides), I get

$$\left([m\partial_t^2 + \gamma\partial_t + \omega_0^2]f(t) = 0\right)^* \rightarrow [m\partial_t^2 + \gamma\partial_t + \omega_0^2]f^*(t) = 0\ . \tag{2.8}$$

   Since $f(t)$ solves the equation, then this procedure tells us is that $f^*(t)$ is also a solution to the same equation so that both $f(t)$ and $f^*(t)$ are solution to the same equation (note that if the coefficients of the equation were complex, the $f^*(t)$ would be a solution to a different equation, one where the coefficients are replaced with their complex conjugates.

2. The equation is linear in $x(t)$. If I have two solutions $f_1(t)$ and $f_2(t)$ to the original equation (i.e. $[m\partial_t^2 + \gamma\partial_t + \omega_0^2]f_1(t) = 0$ and $[m\partial_t^2 + \gamma\partial_t + \omega_0^2]f_2(t) = 0$) then $x(t) = c_1 f_1(t) + c_2 f_2(t)$ is also a solution where c_1 and c_2 are complex numbers (constants):

$$\begin{aligned} & [m\partial_t^2 + \gamma\partial_t + \omega_0^2](c_1 f_1(t) + c_2 f_2(t)) \\ = \ & [m\partial_t^2 + \gamma\partial_t + \omega_0^2]c_1 f_1(t) + [m\partial_t^2 + \gamma\partial_t + \omega_0^2]c_2 f_2(t) && (2.9) \\ = \ & c_1[m\partial_t^2 + \gamma\partial_t + \omega_0^2]f_1(t) + [m\partial_t^2 + c_2\gamma\partial_t + \omega_0^2]f_2(t) && (2.10) \\ = \ & 0 \end{aligned}$$

   This is the principle of superposition discussed in Section 1.6 of Dubin

Now we can write

$$\begin{aligned} x(t) &= e^{-\gamma t/2}[Ae^{i\omega t} + A^* e^{-i\omega t}] \\ &= e^{-\gamma t/2}[(\mathrm{Re}A + i\mathrm{Im}A)e^{i\omega t} + (\mathrm{Re}A - i\mathrm{Im}A)e^{-i\omega t}] \\ &= e^{-\gamma t/2}[\mathrm{Re}A(e^{i\omega t} + e^{-i\omega t}) + i\mathrm{Im}A)(e^{i\omega t} - e^{-i\omega t})] \\ &= e^{-\gamma t/2}[2\mathrm{Re}A\cos(\omega t) - 2\mathrm{Im}A\sin\omega t]\ . \end{aligned} \tag{2.11}$$

However, as emphasized above, it is going to be much easier to work with $e^{\pm i\omega t}$ than $\cos\omega t$ and $\sin\omega t$. You should know how to go between the two, but at some point in the course we will only use complex exponentials.

2.2 Coupled Harmonic Oscillators

A classic mechanics problem is the evolution of two harmonic oscilators that are themselves coupled together by a linear restoring force. A simple example you might have seen is two pendua that are connected by a spring. Now we have two equations

$$m\ddot{x}_1 = -\kappa x_1 - \kappa'(x_1 - x_2) \tag{2.12}$$
$$m\ddot{x}_2 = -\kappa x_2 - \kappa'(x_2 - x_1) \tag{2.13}$$

I have chosen the masses and spring constants to be the same so that it is easy to find the solutions.

To find the solution to these coupled equations, we can try adding and subtracting these two equations. If we define $x_\pm \equiv x_1 \pm x_2$, then adding the two equations gives

$$m\ddot{x}_1 + m\ddot{x}_2 = -\kappa x_1 - \kappa'(x_1 - x_2) - \kappa x_2 - \kappa'(x_2 - x_1) \tag{2.14}$$
$$m\ddot{x}_+ = -\kappa x_+ \tag{2.15}$$

and subtracting gives

$$m\ddot{x}_1 - m\ddot{x}_2 = -\kappa x_1 - \kappa'(x_1 - x_2) + \kappa x_2 + \kappa'(x_2 - x_1) \tag{2.16}$$
$$m\ddot{x}_- = -(\kappa + 2\kappa')x_- \tag{2.17}$$

We see that the equations for x_+ and x_- are decoupled and can therefore be solved independently. The solutions are given by $x_\pm = A_\pm e^{i\omega_\pm t} + \text{c.c.}$ where $\omega_+ = \sqrt{\kappa/m}$ and $\omega_- = \sqrt{(\kappa + 2\kappa')/m}$. Now using $x_1 = (x_+ + x_-)/2$ and $x_2 = (x_+ - x_-)/2$ you can find that x_1 and x_2 are the sum of two frequencies, i.e.

$$x_1 = \frac{1}{2}\left(A_+ e^{i\omega_+ t} + A_- e^{i\omega_- t}\right) + \text{c.c.} \tag{2.18}$$
$$x_2 = \frac{1}{2}\left(A_+ e^{i\omega_+ t} - A_- e^{i\omega_- t}\right) + \text{c.c.} \tag{2.19}$$

The two different modes correspond to two types of motion. E.g. for the coupled pendula, the $+$ solution is where the two pendula swing together in phase so that there distance from each other is unchanged. The $-$ solution is where they are completely out of phase so that approach each other but then are repelled by the spring and get shot apart, only to be pulled back together. Imposing initial conditions at $t = 0$ so that only x_1 is moving (initially) and $x_2 = 0$ corresponds to $A_+ = A_- = A$ which is an equal mixture of these two modes.

The downside of the way we solved the problem is that it requires some cleverness in how we combine equations. If we made the masses and spring constants different it would take more work. If we added more springs it quickly gets out of hand. Instead, it is more useful to turn this into a linear algebra problem:

$$m\ddot{\vec{x}} = -\mathbf{K}\vec{x} \tag{2.20}$$

where

$$\mathbf{K} = \begin{pmatrix} \kappa + \kappa' & -\kappa' \\ -\kappa' & \kappa + \kappa' \end{pmatrix} \tag{2.21}$$

and $\vec{x} = (x_1, x_2)$.

Side Note: $\vec{x}$ is not a really a vector in 2d space, but is instead a vector we invented as a way to keep track of two 1d oscillators. This is important because we could easily imagine making $\vec{x} = (x_1, .., x_{100})$ describing 100 coupled oscillators, e.g. like 100 atoms coupled together. In this way, we can very easily imagine $\vec{x}$ being an extremely large vector (like Avogadro's number big) but we shouldn't confuse this with one harmonic oscillator living in 100-dimensional (or Avagadro's number-dimensional) space, although there may be limits where the solutions are equivalent.

The eigenvalues and eigenvectors of the matrix $\mathbf{K}$ are κ and $\kappa + 2\kappa'$ for $\vec{x}_+ = (1, 1)$ and $\vec{x}_- = (1, -1)$ respectively. I.e.

$$\mathbf{K}\vec{x}_+ = \kappa\vec{x}_+ \qquad \mathbf{K}\vec{x}_- = (\kappa + 2\kappa')\vec{x}_- \ . \tag{2.22}$$

Now suppose that $\vec{x} = A(t)\vec{x}_+$ then our equation is just $\vec{x}_+(\ddot{A}(t) + \kappa A(t)) = 0$. Since $x_+ \neq 0$, $A(t) = e^{i\omega_+ t} + \text{c.c.}$ is our solution. Repeating for $\vec{x} = B(t)\vec{x}_-$ we see that our general solution is

$$\vec{x} = Ae^{i\omega_+ t}\vec{x}_+ + Be^{i\omega_- t}\vec{x}_- + \text{c.c.} \tag{2.23}$$

This is the same solution as the first time, but we see now that our adding and subtracting the equation was just a way of finding the eigenvalues and eigenvectors.

Aside: Linear Algebra: *In a mathematical sense, this course is about extending the ideas of linear algebra from the space of N-dimensional vectors to the space of functions (this sentence will hopefully make sense later). The key idea is that physics is described by function, but the kinds of functions that arise in a lot of physical systems have special properties that make them behave a lot like vectors.*

The key idea is that what we need to understand about linear algebra starts and ends with eigenvector decomposition. Recall that a linear operation, $\hat{L}$, has two basics properties

$$\hat{L}[x + y] = \hat{L}[x] + \hat{L}[y] \qquad \hat{L}[\lambda x] = \lambda\hat{L}[x] \tag{2.24}$$

where x and y are functions and λ is a number. The case you should know well is where $\hat{L} = \mathbf{M}$ is a matrix and $\vec{x}$ and $\vec{y}$ are vectors and $\mathbf{M}\vec{x}$ is matrix multiplication. However, we will see throughout this course that most differential equations you study are also just linear operations (e.g. you can check that both differentiation and integration are both linear operations).

We will get most of our intuition from the eigenvalue decomposition of a matrix. Recall that, given a matrix $\mathbf{M}$, the (left/right) eigenvalues λ_i and (left/right) eigenvectors, u_i, are defined by

$$\mathbf{u}_i^{\mathrm{T}}\mathbf{M} = \lambda_i\mathbf{u}_i^{\mathrm{T}} \qquad\qquad \mathbf{M}\mathbf{u}_i = \lambda_i\mathbf{u}_i \tag{2.25}$$

for the left and right eigenvectors respectively and $^{\mathrm{T}}$ denotes the transpose of the vector. If the matrix is either symmetric or Hermitian, then the right and left eigenvectors / values are the same.

This leads to the first useful result, which is that the eigenvectors for distinct eigenvalues of a symmetry/hermitian matrix are orthogonal. Proof:

$$\mathbf{u}_j^{\mathrm{T}}\mathbf{M}\mathbf{u}_i = \lambda_i\mathbf{u}_j \cdot \mathbf{u}_i = \lambda_j\mathbf{u}_j \cdot \mathbf{u}_i \tag{2.26}$$

where we used defining equation of the right and left eigenvectors in the first and second equality. If $\lambda_i \neq \lambda_j$, this equation only makes sense if $\mathbf{u}_j$ and $\mathbf{u}_i$ are orthogonal, $\mathbf{u}_j \cdot \mathbf{u}_i = 0$. In this sense, we see that find a basis of orthogonal direction and finding eigenvectors are related problems.

Given a $n \times n$ matrix, $\mathbf{A}$, with n-linearly independent eigenvectors, one write the matrix in diagonal form

$$\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1} \tag{2.27}$$

where

$$\mathbf{\Lambda} \equiv \lambda_i \delta_{ij} \tag{2.28}$$

is a diagonal matrix of all the eigenvalues and $Q_i^j = u_i^j$, i.e. the ith column of Q is the eigenvector of eigenvalue λ_i. This works because

$$\mathbf{A}\mathbf{Q} = \mathbf{Q}\mathbf{\Lambda} \tag{2.29}$$

by construction (this is just the eigenvalue equation). Since all the eigenvectors are independent, $\mathbf{Q}$, is an invertable matrix and therefore $\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}$. This way of writting the matrix is very powerful because now

$$\mathbf{A}^n = \prod_{i=1}^{n} (\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}) = \mathbf{Q}\mathbf{\Lambda}^n\mathbf{Q}^{-1} \tag{2.30}$$

where now $\mathbf{\Lambda}^n = \lambda_i^n \delta_{ij}$. In this sense one can even define fractional powers, exponentials, logs, etc. of matrices just using how those functions act on the eigenvalues.

We can immediately see how this would generalize to an arbitrary number of spring with different masses and different spring constants. Suppose we have N springs described by the equation

$$\ddot{\vec{x}} = -\tilde{\mathbf{K}}\vec{x} \tag{2.31}$$

where $\tilde{\mathbf{K}}$ is now an $N \times N$ real matrix. Suppose it has N non-zero eigenvalues λ_i and eigenvectors $\vec{x}_i$ where $i = 1, .., N$. For any eigenvector, we have

$$\tilde{\mathbf{K}}\vec{x}_i = \lambda_i \vec{x}_i \tag{2.32}$$

For each eigenvector, x_i we have a time-dependent coefficient $A_i(t)$ then $\ddot{A}_i(t) + \lambda_i A_i(t) = 0$ such that

$$\vec{x}(t) = \sum_{i=1}^{N} A_i(t)\vec{x}_i \tag{2.33}$$

If all the $\lambda_i > 0$ and real, then we can write

$$\vec{x} = \sum_{i=1}^{N} a_i e^{i\sqrt{\lambda_i}t}\vec{x}_i + \text{c.c.} \tag{2.34}$$

If I give you specific model that corresponds to some specific $\tilde{\mathbf{K}}$, then the solution is found just by calculating these eigenvalues and eigenvectors. For a large matrix, this is itself a difficult problem but one that can but done by computer.

Using our knowledge of linear algebra, we can also notice an amusing fact. Recalling the $\mathbf{M}^n = \mathbf{Q}\mathbf{\Lambda}^n\mathbf{Q}^{-1}$ for integer n, we could go so far as to define

$$\mathbf{M}^{1/2} \equiv \mathbf{Q}[\mathbf{\Lambda}^{1/2}]Q^{-1} \qquad \exp \mathbf{M} = \mathbf{Q}[\exp \mathbf{\Lambda}]\mathbf{Q}^{-1} \tag{2.35}$$

where we define the operation on $\mathbf{\Lambda}$ by acting on all the eigenvalues. Now we can write our general solution as

$$\vec{x} = \mathrm{Re}\left[\exp[it\tilde{\mathbf{K}}^{1/2}]\right]\vec{x}(t=0) + \mathrm{Im}\left[\exp[it\tilde{\mathbf{K}}^{1/2}]\tilde{\mathbf{K}}^{-1/2}\right]\dot{\vec{x}}(t=0)\ . \tag{2.36}$$

It is an exercise for the reader to see that this is equivalent to our first answer. We have used the real and imaginary parts to write the constants in terms of the initial position, $\vec{x}(t=0)$ and velocity $\dot{\vec{x}}(t=0)$. This works because $\exp[it\tilde{\mathbf{K}}^{1/2}]|_{t=0}$ is a the identity matrix which is purely real.

2.3 The driven harmonic oscillator

Now we consider the situation where we drive the system with an external form $F_{\mathrm{ext}} = ma_0\cos(\omega' t)$:

$$\ddot{x}(t) + \gamma\dot{x}(t) + \omega_0^2 x(t) = a_0\cos(\omega' t)\ : \tag{2.37}$$

Now we should be careful about how to make this a complex equation. Note that if we are solving

$$[\partial_t^2 + \gamma\partial + \omega_0^2][x_0 + \mathrm{c.c}] = \frac{1}{2}a_0 e^{i\omega' t} + \mathrm{c.c.} \tag{2.38}$$

then if we can solve

$$[\partial_t^2 + \gamma\partial + \omega_0^2]x_0 = \frac{1}{2}a_0 e^{i\omega' t} \tag{2.39}$$

when we automatically solve the equation. Let's try our ansatz again:

$$Ae^{\rho t}[\rho^2 + \rho\gamma + \omega_0^2\rho] = \frac{1}{2}a_0 e^{i\omega' t} \tag{2.40}$$

First of all, we must have $\rho = i\omega'$ if we have any hope of solving this equation at every t. Solving for A we get

$$A = \frac{1}{2}\frac{a_0}{-\omega'^2 + i\gamma\omega + \omega_0^2} = \frac{a_0}{2}\frac{(\omega_0^2 - \omega'^2) - i\omega'\gamma}{(\omega_0^2 - \omega'^2)^2 + (\omega'\gamma)^2} \tag{2.41}$$

and therefore

$$x(t) = x_{\mathrm{homogenous}} + \frac{a_0}{(\omega_0^2 - \omega'^2)^2 + (\omega'\gamma)^2}\left[(\omega_0^2 - \omega'^2)\cos\omega t + \omega\gamma\sin\omega' t\right] \tag{2.42}$$

We see the usual intuition that if we drive the system near the natural frequency $\omega' = \omega_0$, the we get a large amplitude oscillation in response. If we assume $\gamma\omega > |\omega_0^2 - \omega'^2|$ then

$$x(t) \approx \frac{a_0}{\omega_0\gamma}\sin\omega_0 t \tag{2.43}$$

which is parametrically larger than the force itself when $\omega_0\gamma \ll a_0$. I.e. when the damping is very small, we can fine tune the frequency so that a small external force produces very large response.

2.4 Summary

Why did we learn all this right now? We had to remember complex numbers and linear algebra to solve some seeming random problem about coupled harmonic oscillators. The key insight that came from linear algebra is that life is much easier when we can replace matrix multiplication with the eigenvalue:

$$\tilde{\mathbf{K}}\vec{x}_i = \lambda_i \vec{x}_i \ . \tag{2.44}$$

In fact, we had already pulled the same trick with the derivatives with our ansatz:

$$\frac{d}{dt}e^{i\omega t} = i\omega e^{i\omega t} \ . \tag{2.45}$$

This is a general lesson: *the laws of physics makes way more sense if you can replace complicated mathematical operations with numbers (or simple functions).* If you internalize this idea now with these examples, it will be much easier to process everything that is coming up this quarter. The purpose of this course is to generalize this idea far beyond the point you may have realized is possible.

3 The Wave Equation and the Fourier Transform

Dubin Chapter 2

3.1 Review

At some point, everyone should have seen the wave equation:

$$\frac{\partial^2\phi(x,t)}{\partial t^2} - c_s^2\frac{\partial^2\phi(x,t)}{\partial x^2} = 0 \ . \tag{3.1}$$

Given any function of space, $g(x)$ and the boundary condition $\phi(t=0,x) = g(x)$, this equation has solutions

$$\phi = ag(x + c_s t) + bg(x - c_s t) \tag{3.2}$$

where $a + b = 1$.

A priori, this looks like an intimidating equation. Had we not guessed the form of the solution, it might not have been obvious that we could derived the solution so easily. After all, this is a partial differential equation and we don't have many general tools for such situations.

From your previous experience, you probably also have a sense that waves are about oscillations moving in space and time (think of a wave pattern on the surface of the ocean), but our solution is true of literally any shape we can imagine. The reason is that most physical waves are a bit more complicated than this equations suggests. The picture of oscillations emerges as we try to understand where these solutions come from in a way that works for all-kinds of waves in the real world and not just the ideal waves.

We can get a bit of intuition if we assume that $\phi(t=0,x) = Ae^{ikx} + \text{c.c.}$. Let us guess that there is a solution of the form $\phi(t,x) = f(t)e^{ikx} + \text{c.c.}$. The equation for $f(t)$ becomes:

$$[\ddot{f} + c_s^2k^2f]e^{ikx} = 0 \rightarrow [\ddot{f} + c_s^2k^2f] = 0 \ . \tag{3.3}$$

This is nothing other than the simple harmonic oscillator we solved last time. From our past experience, we know $f(t) = ae^{ic_skt} + be^{-ic_skt}$ so that

$$\phi(t,x) = ae^{ik(c_st+x)} + b^*e^{ik(c_st-x)} + \text{c.c.} \tag{3.4}$$

Notice that we get the $c_st \pm x$ structure again, but this time we got it just from our knowledge of the harmonic oscillator. By making this assumption about the initial shape, we found the equation takes a form that we have lots of tools to handle and we can derive the result directly. However, you might feel this is a bit of a false achievement: I put in a special solution and I found a special solution. What are we going to do when we face new equations and we don't know what guesses or special solutions to look for?

The new idea that brings real power to this and many other problems is *the Fourier transform.* The first step is just like the initial guess I made before: imagine I chose the initial shape so that the solution could take the form $\phi = ae^{i\omega t}e^{ikx}$ for some ω and k. In this case my equation becomes:

$$[\omega^2 - c_s^2k^2]a = 0 \tag{3.5}$$

This is amazing, I started with a PDE and I have reduced it to an algebraic equation. Now our solution is just $\omega = \pm c_s k$! In fact, we can see this trick would work for any equation that is made only of derivatives of t and x, no matter how complicated: it would always reduce to a polynomial equation!

Aside: It is also worth noting that our exponentials are "eigenvectors" of the derivative operation, i.e.

$$\frac{\partial}{\partial t} e^{i\omega t} = i\omega e^{i\omega t} \ . \tag{3.6}$$

The derivative is a linear operator so if we thing of $\partial_t = \hat{L}$ then $e^{i\omega t}$ in an eigenvector of $\hat{L}$ with eigenvalue $i\omega$. In this sense, we can also see that what we are trying to do is decompose the differential equation in terms of the eigenvectors of the differential operator that defines the equation $\hat{L}[f(x)] = 0$.

Physically, there is a very good reason to think in terms of frequencies and wavelengths instead of positions and times. We can see this using some basic facts about light that you already know very well. When light travels through a medium, we know that the speed of propagation depends on the color of light. When we think in ω and k, this is no big deal: I just have to solve

$$\omega^2 - c_s^2(\omega) k^2 = 0 \tag{3.7}$$

This just slightly changes the relationship between k and ω at any given frequency. A lot of very important aspects of optics are encoded in this seeming innocuous statement.

In contrast, for some generic function $c_s^2(\omega)$, you might not have any idea how you would even write a differential equation that gives this behavior. We will see how to do it but we will also see that this seemingly simple generalization of the usual wave equation is a complicated mess when you try to write it as a differential equation.

From this simple example, we can see how our physical intuition lives in the space of frequencies and wavelengths not positions and times. The color of light is just a statement of its angular frequency ω or its wavenumber k. At this point, you are likely all familiar with the idea that light can be decomposed into the individual colors (using a prism or example) to make a rainbow (in the case of sunlight).

The Fourier transform is, very literally, the decomposition of light into its individual colors; however, instead of using a prism, we will define it as a mathematical procedure. While either is good for light, this mathematical procedure will let us decompose anything into colors / frequencies / waves, whether it physically wants to treat them individually or not. Moreover, just as the waveform of light can take the shape of any (reasonably smooth) function (i.e. any $g(x)$ as appears in Equation (3.2)), we must be able to decompose any function in the wavelengths and frequencies (colors), regardless of whether it is a light, sound, or literally any other kind of information you could imagine.

3.2 The Fourier Series

Dubin Chapter 2.1,2.2

At this point, it may still seem that we are still talking about specialized examples. Now

comes the truly brilliant and profound part: we can break any[2] function into a sum of these types of oscillations without making an error! The precise statement is the following (via the textbook):

Theorem: If a function $f(x)$ is continuous on $x \in (-L, L)$ and its derivative is non-infinite and sectionally continuous, then you can write a Fourier series such that is equal to $f(x)$ at each point. In other words, for any function $f(x)$ obeying these conditions, there exists a fourier series representation

$$f_{\text{FS}}(x) = \sum_{n=0}^{\infty} a_n \cos(\pi n x/L) + b_n \sin(\pi n x/L) \tag{3.8}$$

such that $f_{\text{FS}}(x) = f(x)$ for every $x \in (-L, L)$.

I will leave it to the math department to prove this statement (and its limitations). Instead, it is easier to see how it works in action. Let us consider a function on the space $[\pi, \pi]$ with periodic boundary condition (i.e. $f(x \pm 2\pi) = f(x)$). In order for these boundary conditions to work, we need to pick the frequencies of the sines and cosines to have the same periodicity: $\sin(nx)$ and $\cos(nx)$ where $n \in \mathbb{Z}$ so that $\sin(n(x \pm 2\pi)) = \sin(nx + n2\pi) = \sin(nx)$. Let us suppose this works and we can write

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(nx) + b_n \sin(nx) \tag{3.9}$$

for some coefficients a_n and b_n (I dropped the f_{FS} because the theorem tells us that these functions are the same). Part of the magic in knowing that such a representation exists is that it gives us a recipe to figure out what these coefficients are. To see how this works, let start by noticing what happens when I integrate both sides of this equation:

$$\begin{aligned}
\int_{-\pi}^{\pi} dx f(x) &= \sum_{n=0}^{\infty} \int_{-\pi}^{\pi} dx a_n \cos(nx) + b_n \sin(nx) \\
&= 2\pi a_0 + \sum_{n=1}^{\infty} n^{-1} [a_n \sin(nx) - b_n \cos(nx)]|_{-\pi}^{\pi} \\
&= 2\pi a_0
\end{aligned} \tag{3.10}$$

Just be integrating, we have isolated a single term in the sum! The intuition here is pretty simple: every term in the sum is an oscillation around 0, so when we average we are just canceling to the positive and negative parts of the oscillation. Now suppose we tried another integral:

$$\begin{aligned}
\int_{-\pi}^{\pi} dx \sin(3x) f(x) &= \sum_{n=0}^{\infty} \int_{-\pi}^{\pi} dx a_n \sin(3x)\cos(nx) + b_n \sin(3x)\sin(nx) \\
&= \sum_{n=0}^{\infty} \int_{-\pi}^{\pi} dx \left(\frac{1}{2} a_n [\sin((n+3)x) + \sin((n-3)x)] \right. \\
&\qquad \left. + \frac{1}{2} b_n [\cos((n-3)x) - \cos((n+3)x)] \right)
\end{aligned} \tag{3.11}$$

[2] "Any" in the sense of function describing physical phenomena, but not all functions in the mathematical sense.

We have used trig identities to rewrite this as something like the sum we had before. If we pick a random n then we expect all of these oscillations to cancel out and we will get zero. But for the special case of $n = 3$ we have $\cos((n-3)x) \to 1$ and we get

$$\int_{-\pi}^{\pi} dx \sin(3x) f(x) = \pi b_3 \tag{3.12}$$

How did this happen? The key is to notice that when we write just the $n = 3$ part of the integral, we are integrating over $b_n[\sin 3x]^2 = b_n \frac{1}{2}[1 - \cos 6x]$. Clearly $[\sin 3x]^2$ is positive everywhere so there is no way the integral could vanish. The miraculous thing is that we again isolated just a single term in the sum and that there or no other contributions that behave this way. From here, it is easy to see the pattern:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx f(x) \tag{3.13}$$

$$a_{n>0} = \frac{2}{2\pi} \int_{-\pi}^{\pi} dx \cos(nx) f(x) \tag{3.14}$$

$$b_n = \frac{2}{2\pi} \int_{-\pi}^{\pi} dx \sin(nx) f(x) \tag{3.15}$$

Now we have the tools to take any function and write it as a fourier series.

I wrote the Fourier transform in terms of sines and cosines to make it easier to understand. But, as usual, it will be advantageous to work in terms of complex exponentials as well. So let us take the series and write is as

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n \cos(nx) + b_n \sin(nx) \\ &= \frac{1}{2}\left[\sum_{n=0}^{\infty} (a_n + ib_n) e^{-inx} + \sum_{n=0}^{\infty} (a_n - ib_n) e^{inx}\right] \\ &= \sum_{n=-\infty}^{\infty} c_n e^{-inx} \end{aligned} \tag{3.16}$$

where $c_n = (a_n + ib_n)/2$ for $n \geq 0$ and $c_n = (a_{-n} - ib_{-n})/2$ for $n < 0$. The fact $c_n^* = c_{-n}$ is what encodes that $f(x)$ is a real quantify, i.e. $(f(x))^* = f(x)$. We can see this directly from the series

$$(f(x))^* = \sum_{n=-\infty}^{\infty} c_n^* e^{inx} = f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-inx} \to c_n^* = c_{-n} \ . \tag{3.17}$$

This representation is slightly more compact notation in that we have extended the sum to $n \in \mathbb{Z}$ but now with only a single term for each n in the sum. Now we should also have a simplified formula for the c_n, right? First, we need the orthogonality relation

$$\int_{-\pi}^{\pi} dx e^{imx} e^{inx} = 2\pi \delta_{-n,m} \tag{3.18}$$

(we used $e^{in\pi} = e^{2in\pi} e^{-in\pi} = e^{-in\pi}$). So we can write

$$c_n = \frac{1}{2\pi} \int dx e^{inx} f(x) \tag{3.19}$$

Notice that this formula simplifies what was an odd looking factor of 2 in the cosine / sine description. Specifically, the formula for a_0 was different from $a_{n>0}$ and b_n by a weird factor of two. The reason was that when we average $\sin^2$ or $\cos^2$ we get a factor of $1/2$ that doesn't exist for a_0. Using complex numbers, this went away because we are averaging over $e^{i(n-n)x)} = 1$ for every n including $n = 0$.

This mysterious factor of two survives in the full sum if we remember that $c_n* = c_{-n}$ so when we sum over n, we are getting the sum of c_n and its complex conjugate, *except* for $n = 0$ where there is one one term in the sum. So $n = 0$ is still special but now in a way that is a bit more obvious.

Finally, let us recall why we started doing this: by replacing a function with an exponential, we replace a differential equation (with constant coefficients) with a algebraic equation. We can solve the equation for each term in the fourier series so we write

$$f(x,t) = \sum_n a_n e^{-i\omega_n t - inx} \rightarrow \omega_n = \pm c_s n \tag{3.20}$$

so that our full solution is now

$$f(x,t) = \sum a_n e^{-in(c_s t - x)} + b_n e^{-in(c_s t + x)} + \text{c.c.} \tag{3.21}$$

This is nothing more than the Fourier transform of the general solution. But now we have found the same solution using a tool that we can apply to a much wider range of problems.

3.3 A Simple Example

Having established the power of this method, we should understand what the Fourier transform is all about, independent of our initial motivations. To make this more concrete, let's work out a simple example. Suppose we have a function on $[-\pi, \pi]$ defined as

$$f(x < 0) = 1 + x/\pi \qquad\qquad f(x \geq 0) = 1 - x/\pi \ . \tag{3.22}$$

(I have chosen $f(x = -\pi) = f(\vec{x} = +\pi)$ so that it is consistent with periodic boundary conditions without breaking continuity.). Since $f(-x) = f(x)$ but $\sin n(-x) = -\sin nx$, it is easy to see that $b_n = 0$ for all n. So all we need to calculate is

$$\begin{aligned} a_n &= \frac{2}{2\pi}\int_{-\pi}^{\pi} dx f(x)\cos nx & (3.23)\\ &= \frac{4}{2\pi}\int_0^{\pi} dx(1 - x/\pi)\cos nx & (3.24)\\ &= \frac{2 - 2\cos n\pi}{n^2\pi^2} & (3.25) \end{aligned}$$

For $n = 0$ we get $a_0 = 1/2$.

Aside: I always forget how to do these integrals analytically, but here is a trick. First let's split the two terms:

$$\int_0^{\pi} dx(1 - x/\pi)\cos nx = \int_0^{\pi} dx \cos nx - \int_0^{\pi} dx(x/\pi)\cos nx \ . \tag{3.26}$$

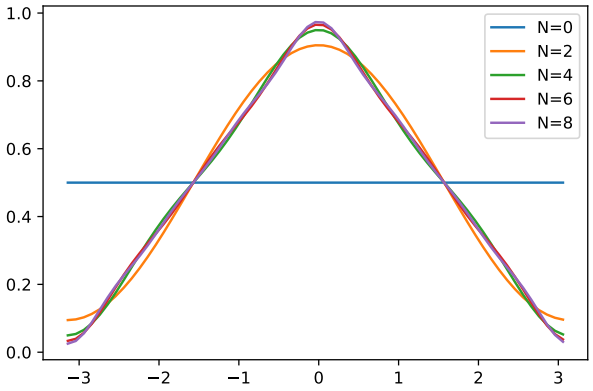

Figure 2: The fourier series for $f(x)$ defined in (3.22) if we keep only up to $n = N$. We see that it converges fairly quickly to the original function.

The first term vanishes as $\int_0^\pi dx \cos nx = n^{-1} \sin nx|_0^\pi = 0$. *We can use a trick for the second term*

$$-\int_0^\pi dx(x/\pi)\cos nx = \frac{1}{\pi}\frac{d}{dn}\int_0^\pi dx \sin nx = \frac{1}{\pi}\frac{d}{dn}\frac{\cos(n\pi)-1}{n} \tag{3.27}$$

Noticing tha $\frac{d}{dn}\cos n\pi = -\sin n\pi = 0$, *we get*

$$-\int_0^\pi dx(x/\pi)\cos nx = -\frac{1}{\pi}\frac{1}{n^2}(\cos n\pi - 1) \tag{3.28}$$

In Figure 2, we have the result of this calculation when we define

$$f_N = \sum_{n=0}^{N} a_n \cos nx \ . \tag{3.29}$$

Clearly keeping only $n = 0$ is a bad idea as it just gives the average. Since $a_1 = 0$, the first non-trivial term is a_2 which gives a pretty good approximation. We see that we don't need very many terms to reproduce the function quite well.

Let us consider another example, but now we will use the complex exponential representation. Suppose have the function $f(x) = x^2$ defined on $[-\pi, \pi]$ with period boundary conditions. This function is continous, even at the boundaries because $f(-\pi) = f(+\pi) = \pi^2$. Coefficients of the Fourier series can be calculated as follows. For $n = 0$, we have

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx x^2 = \frac{1}{3}\pi^2 \tag{3.30}$$

and for $n \neq 0$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx x^2 e^{inx} \tag{3.31}$$

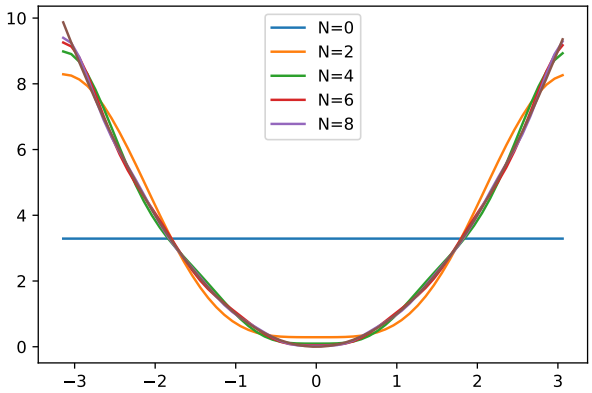

Figure 3: The fourier series for $f(x) = x^2$if we keep only up to $n = N$. We see that it converges fairly quickly to the original function.

Let's remember how to evaluate this integral by hand.

$$\begin{aligned}
\int_{-\pi}^{\pi} dx x^2 e^{inx} &= \int_{-\pi}^{\pi} dx x^2 (in)^{-2} \frac{d^2}{dx^2} e^{inx} && (3.32)\\
&= x^2 (in)^{-2} \frac{d}{dx} e^{inx}|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} dx 2x (in)^{-2} \frac{d}{dx} e^{inx} && (3.33)\\
&= x^2 (in)^{-1} e^{inx}|_{-\pi}^{\pi} - 2x(in)^{-2} e^{inx}|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} dx 2 (in)^{-2} e^{inx} && (3.34)\\
&= x^2 (in)^{-1} e^{inx}|_{-\pi}^{\pi} - 2x(in)^{-2} e^{inx}|_{-\pi}^{\pi} + 2(in)^{-3} e^{inx}|_{-\pi}^{\pi} && (3.35)\\
& && (3.36)
\end{aligned}$$

Noticing now that $e^{in\pi} = e^{-in\pi}$ for any integer n, we see that the first and last terms vanish and we are left with

$$\int_{-\pi}^{\pi} dx x^2 e^{inx} = \frac{4\pi}{n^2}(-1)^n \tag{3.37}$$

so that

$$c_n = \frac{2}{n^2}(-1)^n \tag{3.38}$$

Notice that this is purely real, so that when we write the Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-inx} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos(nx) \tag{3.39}$$

where used the fact c_n is real to write the answer in terms of $2\cos(nx)$. The comparison of the series with the original function is shown in Figure 3.

3.4 Broader Applications and Limitations

We were lead to the fourier transform from the wave equation: each mode evolves independently and its time evolution is determined by a algebraic equation. However, the fourier series is a part

of a much broader idea here which is that we can replace some object, in this case $f(x)$, with a sum over some set of normal / basis objects.

The canonical example here is vectors: you have some vector $\vec{v}$ and a normal basis $\hat{x}$, $\hat{y}$ and $\hat{z}$. You can always write $\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$. Furthermore, since I will choose $\hat{x} \cdot \hat{x} = 1$ and $\hat{x} \cdot \hat{y} = \hat{x} \cdot \hat{z} = 0$, I can also determine these coefficients as

$$\vec{v} \cdot \hat{x} = v_x \ . \tag{3.40}$$

The fourier transform is the same exact idea, except now we are representing the space of functions on the region $[-\pi, \pi]$ in terms of an infinite set of basis functions and the dot product has become an integral

$$f \cdot \hat{\cos}_n \equiv \int dx f(x) \cos nx \ . \tag{3.41}$$

The common feature is that we are using the "orthogonality" ($\int dx \cos nx \cos mx = \delta_{nm}$) to isolate the individual components.

This idea of thinking of the space of functions as some kind of vector space is incredibly useful. In physics, this idea is at the heart of quantum mechanics, where the functions that make up this basis are typically individual quantum states of constant energy. Beyond physics, this is commonly used method to analyze data, especially noisy data. In particular, noise can effect some of these modes more than others (e.g. noise can be dominated by high or low frequencies) and thus by decomposing the data into such a basis, you can more easily eliminate the noise without losing signal.

I made a few comments about the types of functions for which the Fourier transform can fail, but the reason might not seem intuitive. After all, I have provided a recipe to compute a_n and b_n for any function, as long as it is integrable, which is a much weaker condition that continuity. To see the problem, consider the function $f(x < 0) = -1$ and $f(x > 0) = 1$ and $f(x = 0) = f_0$. First one can see that $a_n = 0$ for all n. Iff we had a function $g(x) = 0$ for all $x \neq 0$ and $g(0) < \infty$, then $\int dx g(x) \cos nx = 0$. In particular, this means that the formula for a_n doesn't depend on $f(x = 0)$. As a result, since $\cos(-nx) = \cos nx$ but $f(-x) = -f(x)$ it must be the case that $a_n = 0$ for all n. Therefore our fourier series is

$$\tilde{f}(x) = \sum b_n \sin(nx) \tag{3.42}$$

but clearly this has the property that $\tilde{f}(x = 0) = 0$. Therefore $\tilde{f}(x) \neq f(x)$ unless $f_0 = 0$.

This may seem like a minor issue, as we are talking about the disagreement between the original function and its fourier transform at a single point. For the purpose of physics, this is indeed a relatively minor point, as we are often not interested in exactly how such a function would behave at $x = 0$. In cases where we use a discontinuous function, we are usually approximating a smooth function that changes quickly over a short period of time. However, there is a principle here that does have applications elsewhere. What we noticed here is that if $g(x) = 0$ for all $x \neq 0$ and $|g(x = 0)| < \infty$, then

$$\int dx g(x) \sin nx = \int dx g(x) \cos nx = 0 \tag{3.43}$$

$$a_n = b_n = 0 \tag{3.44}$$

for all n. In other words, there are a set of functions that are indistinguishable by the fourier series from the function $g(x) = 0$.

So what does it mean that there are a set of functions $g(x)$ where these integrals always vanish? Imagine you were working with 2-dimensional vectors in the basis $\hat{x}$ and $\hat{y}$ and someone came along and told you that there was a new kind of vector $\hat{z} \neq 0$ such that $\hat{z} \cdot \hat{x} = \hat{z} \cdot \hat{y} = 0$. The way that I have named this suggests that you have discovered that $\hat{z}$ is really a new dimension and that you need to expand you basis to capture it.

With functions, it is a bit more confusing. In some sense, it is an infinite dimensional space, but it still makes sense to ask if your basis ($\cos nx$ and $\sin nx$) is complete. For continuous, twice differentiable functions, the answer is that the fourier basis is complete. Every such function is perfectly represented in terms of this basis. We have now seen that if we don't satisfy these criteria, it is not complete: there are functions that are non-zero but have zero "length" this basis.

The reason I am mentioning this is that there are lots of ways to break a function into a sum of other functions. We will discuss a few in this course and you may encounter more in your lives beyond this course. E.g. you could pick a set of polynomials, $p_n(x)$ such that $\int dx p_n(x) p_m(x) = \delta_{nm}$. The most important[3] question to ask when picking such a basis of functions is whether it is "complete" in the sense of the problem you are interested in. I.e. are the types of objects you are using fully capture by the basis you are using.

Aside: Gibbs Phenomenon

This section is option reading that we will note cover in class

Another and perhaps more serious issue with discontinuous functions is the convergence of the fourier series near the discontinuous point. The easiest thing we can do is consider a square wave $f(x) = -1$ for $x < 0$ and $f(x) = 1$ for $x > 0$ (imposing periodic boundary conditions makes this a square wave). Again, this is an odd function so $a_n = 0$ for all n and

$$b_n = \frac{2}{\pi} \int_0^\pi dx \sin nx = \frac{2}{n\pi}(1 - (-1)^n) \tag{3.45}$$

Using this in the series we have

$$f(x) \approx \sum_{n=1,3,5,\ldots} \frac{4}{n\pi} \sin(nx) \tag{3.46}$$

The result is shown in Figure 4. Notice the values of N used in this plot: $N = 20, 50, 100$. In the previous example, we did quite well with $N = 8$. Furthermore, as we increase N, it doesn't seem to be doing much for the overshoot on the corners of the square wave. This slow convergence near a discontinuity is known as Gibbs phenomenon.

To see the problem, let us consider the series with N terms such that

$$f_N(x) = \sum_{n=1,3,5}^{N} \frac{4}{n\pi} \sin(nx) \tag{3.47}$$

[3]The second most important question to ask yourself is whether the series description is practical in the sense that can get a pretty good description with a few terms. E.g. if I am looking at functions that oscillate a lot, then it is going to take a lot of terms in a polynomial to capture it, but it might take only one or two fourier coefficients.

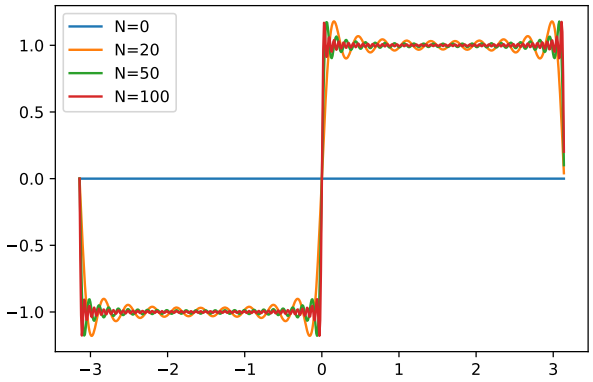

Figure 4: Gibbs phenomenon.

but now let's evaluate it at the point $x = \pi/N$*:*

$$f_N(\pi/N) = \sum_{n=1,3,5}^{N} \frac{4}{n\pi} \sin(n\pi/N) \tag{3.48}$$

Remember a Reimann sum is related to an integral by

$$\int dx h(x) = \lim_{\Delta x \to 0} \Delta x \sum_n h(n\Delta x) \tag{3.49}$$

Let us first consider just the sum $\sum_{n=1} \frac{1}{n}$*. In this case we can think of this as*

$$\sum_{n=1}^{\infty} \Delta x \frac{1}{n\Delta x} \to \int_{\Delta x}^{\infty} \frac{dx}{x} \to \log x|_{\Delta x}^{\infty} \to \infty \tag{3.50}$$

So $1/n$ *is not going to zero fast enough for this series to converge. Now let us rewrite our actual sum*

$$\begin{aligned} f_N(\pi/N) &= \sum_{m=0}^{N/2} \frac{4}{N} \left[\frac{1}{\frac{\pi}{N} + \frac{2m\pi}{N}} \sin(\frac{\pi}{N} + \frac{2m\pi}{N}) \right] && (3.51) \\ &= 2\Delta x \sum_{m=0}^{N/2} \frac{\sin(\pi/N + \pi m \Delta x)}{(\pi/N + \pi m \Delta x)} && (3.52) \\ &= 2 \int_0^1 dy \frac{\sin \pi x}{\pi x} && (3.53) \end{aligned}$$

where we defined $\Delta x = 2/N$*. Evaluating this integral we find*

$$f_N(\pi/N) = 1 + 2 \times 0.089489\ldots \tag{3.54}$$

where this number $0.089489\ldots$ *is the Wilbraham–Gibbs constant that appears in every such example and* $2 = f(0+\epsilon) - f(0-\epsilon)$ *which is size of the discontinuity. Let us rephrase what happens:*

at any finite N, I can find a point π/N away from the discontinuity such that the error is always the size of the discontinuity times this constant. What this means is that at any finite N, I can find a point where the function $f(x)$ is continuous but the error is about 9 percent of the size of the discontinuity. This is an example non-uniform convergence. While it is true that for every point, there is an N large enough so that it converges to the answer, there is no single N that works for every point.

Notice that this wouldn't happen if the fourier series when to zero like a larger power of n. E.g. suppose instead that our series took the form

$$f_N(\pi/N) = \sum_{n=1}^{N} \frac{4}{(n\pi)^2} \sin(n\pi/N) \tag{3.55}$$

Now suppose we try to write it as a Riemann sum with $\Delta x = \pi/N$, the our Riemann sum would be

$$\begin{aligned} f_N(\pi/N) &\approx \frac{1}{N^2} \sum_{n=1}^{N} \frac{1}{(n\pi/N)^2} \sin(n\pi/N) && (3.56) \\ &\approx \frac{\Delta x}{\pi N} \sum_{n=1}^{N} \frac{1}{(n\Delta x)^2} \sin(n\Delta x) && (3.57) \\ &\approx \frac{1}{\pi N} \int_{\Delta x}^{\pi} dx \frac{\sin x}{x} \to 0 && (3.58) \end{aligned}$$

But this vanishes as $N \to \infty$ so that $f_N(\pi/N) \to 0$. This makes sense because if the function is continuous and odd, the $f(x \to 0) = 0$.

4 Fourier Transforms on the Real Line

Dubin Chapter 2.3

4.1 Extending to Infinite Range

I motived the Fourier transform from the wave equation, but then I focused on the domain $[-\pi, \pi]$. In many applications (e.g. light), we want to consider the domain $[-\infty, \infty]$. On this domain, the Fourier transform also has a clear advantage over some polynomial basis of functions (for example). In particular, because of the oscillations, the orthogonality property easy works on any domain.

First let us consider the domain $[-L, L]$. We can just get this from the previous case, by defining $x = yL/\pi$ such that $y \in [-\pi, \pi]$ so that we have

$$f(y) = \sum_{n=-\infty}^{\infty} c_n e^{-iny} \rightarrow f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-i\pi nx/L} \tag{4.1}$$

Similarly, we can compute the coefficients of this series as

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} dy e^{iny} f(y) = \frac{1}{2L} \int_{-L}^{L} dx e^{in\pi x/L} f(x) \ . \tag{4.2}$$

Now we want to take the limit $L \rightarrow \infty$. To make sense of this limit, we define $k = \pi n/L$ such that $k \in [-\infty, \infty]$. This starts to look like a Reimann sum where $\Delta k = \pi/L \rightarrow 0$.

$$\sum_{n=-\infty}^{\infty} c_n e^{-i\pi nx/L} \rightarrow \lim_{\Delta k \rightarrow 0} \alpha \Delta k \sum_{n=-\infty}^{\infty} \frac{c_n}{\alpha \Delta k} e^{-in\Delta kx} \tag{4.3}$$

where α is some normalization we will pick later (for now it is arbitrary). Now remember a Reimann sum is related to an integral by

$$\int dx h(x) = \lim_{\Delta x \rightarrow 0} \Delta x \sum_n h(n\Delta x) \tag{4.4}$$

then we define

$$f(x) = \alpha \int_{-\infty}^{\infty} dk \tilde{f}(k) e^{-ikx} \tag{4.5}$$

where $\tilde{f}(n\Delta k) = c_n/(\alpha \Delta k)$. So how do we pick α? It is totally arbitrary because I am just changing the definition of $\tilde{f}(k)$. But, remember what makes this expansion useful is that I can isolate individual terms by integrating, so let's see how that works. Using the formula for c_n we have

$$\tilde{f}(n\Delta k) \quad = \quad \frac{1}{2\alpha \Delta k L} \int_{-\infty}^{\infty} dx e^{in\Delta kx} f(x) \tag{4.6}$$

using $n\Delta k = k$ and $\Delta k L = \pi$ we have

$$\tilde{f}(k) = \frac{1}{2\pi\alpha} \int_{-\infty}^{\infty} dx e^{ikx} f(x) \tag{4.7}$$

We will choose $\alpha = 1/2\pi$ so that

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx e^{ikx} f(x) \tag{4.8}$$

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{f}(k) e^{-ikx} \tag{4.9}$$

Be aware, sometimes people choose $\alpha = 1/\sqrt{2\pi}$ so that these formulas look a bit more symmetric in terms of the normalization.

At face value these formulas are pretty difficult to understand without using the Riemann sum to understand what's going on. However, if we invent a new type of object our life becomes a lot easier. The invention is due to Dirac and we will just state it this way

$$\int_{-\infty}^{\infty} dx e^{ikx} = (2\pi)\delta(k) \tag{4.10}$$

where $\delta(k)$ is the "Dirac delta function". I put quotes because it isn't really a function in the mathematical sense. In particular, it doesn't make much sense as a function because it is defined to be $\delta(k) = 0$ for $k \neq 0$ and $\delta(0) = \infty$. Instead, what defines it is how it behave in an integral. In particular:

$$\int dk \tilde{f}(k)\delta(k) = \tilde{f}(0) \tag{4.11}$$

for any smooth function $\tilde{f}(k)$. In fact, we can even define derivatives of $\delta(k)$ using integration by parts:

$$\int dk \tilde{f}(k) \frac{d}{dk}\delta(k) = -\frac{d}{dk}\tilde{f}(k)|_{k=0} \ . \tag{4.12}$$

Notice this means that

$$\int_{-\infty}^{\infty} dk \delta(k) = 1 \ . \tag{4.13}$$

You can think of the δ-function as the limit of a functions with a finite limit and height but I make it more and more narrow but increase the height to keep the area fixed to 1.

To see the utility of this function, we can re-derive our formula for $\tilde{f}(k)$ as

$$\begin{aligned}
\int_{-\infty}^{\infty} dx e^{ikx} f(x) &= \int_{-\infty}^{\infty} dx e^{ikx} \int_{-\infty}^{\infty} \frac{dk'}{2\pi} \tilde{f}(k') e^{-ik'x} \\
&= \int_{-\infty}^{\infty} \frac{dk'}{2\pi} \tilde{f}(k') \int_{-\infty}^{\infty} dx e^{i(k-k')x} \\
&= \int_{-\infty}^{\infty} \frac{dk'}{2\pi} \tilde{f}(k')(2\pi)\delta(k-k') \\
&= \tilde{f}(k)
\end{aligned} \tag{4.14}$$

So now we can understand how we go back and forth between $f(x)$ and $\tilde{f}(k)$ without having to take the limit starting with a finite domain.

Of course, the δ-function can also appear in position when we invert the fourier transform. In particular, suppose $\tilde{f}(k) = 1$

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{(2\pi)} e^{-ikx} = \delta(x) \tag{4.15}$$

In fact, if $\tilde{f}(k)$ is just a polynomial k

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} \frac{dk}{(2\pi)} e^{-ikx} [c_0 + c_1 k + c_2 k^2 + \ldots] \\ &= [c_0 + ic_1 \frac{d}{dx} - c_2 \frac{d^2}{dx^2} + \ldots] \int_{-\infty}^{\infty} \frac{dk}{(2\pi)} e^{-ikx} \\ &= [c_0 + ic_1 \frac{d}{dx} - c_2 \frac{d^2}{dx^2} + \ldots] \delta(x) \end{aligned} \tag{4.16}$$

In other words, anything that looks like a polynomical in fourier space is related to a pure δ-function in position space (and vice versa). We will return to this when we talk about complex integration.

4.2 Some Important Examples

We are in a position to derive some of the extremely simple but useful properties of the Fourier transform. Let's start by just listing a few obvious ones:

- **Linearity:** If $f(x) = g(x) + h(x)$ then $\tilde{f}(k) = \tilde{g}(k) + \tilde{h}(k)$ (assuming that both $\tilde{g}(k)$ and $\tilde{h}(k)$ are well-defined. Obviously if $g(x) = f(x) + \infty$ and $h(x) = -\infty$ this wouldn't make much sense.). This is just a statement that $\int dx[g(x) + h(x)] = \int dx g(x) + \int dx h(x)$.

- **Translations:** if $f(x) = g(x + c)$ then $\tilde{f}(k) = e^{-ikc}\tilde{g}(k)$. This just follows from the definition.

- **Convolution:** Another very useful property we will use over and over again is that

$$f(x) = \int_{-\infty}^{\infty} dx' g(x') p(x - x') \tag{4.17}$$

$$\begin{aligned} \rightarrow \tilde{f}(k) &= \int_{-\infty}^{\infty} dx e^{ikx} \int_{-\infty}^{\infty} dx' g(x') p(x - x') \\ &= \int_{-\infty}^{\infty} dx dx' e^{ikx} \int_{-\infty}^{\infty} \frac{dk_1 dk_2}{(2\pi)^2} e^{-ik_1 x'} e^{-ik_2(x-x')} \tilde{g}(k_1)\tilde{p}(k_2) \\ &= \int_{-\infty}^{\infty} dx \frac{dk_1 dk_2}{(2\pi)} \delta(k_1 - k_2) e^{ikx} e^{-ik_2 x} \tilde{g}(k_1)\tilde{p}(k_2) \\ &= \tilde{g}(k)\tilde{p}(k) \end{aligned} \tag{4.18}$$

  This is amazingly useful! A convolution (often written as $g(x) \star p(x) \equiv \int dx' g(x') p(x - x')$) in position space is just multiplication in Fourier space (and vise versa).

- **Derivatives and Eigenfunctions:** At many points, we have emphasized that the fourier transform, is like expanding in "eigenvectors" of the derivative, given that $\partial_x e^{-ikx} =$

$(-ik)e^{ikx}$. Since these are functions, not vectors, we will given them the name *eigenfunctions*. The particular consequence of this statement is the following: if $g(x) = \partial_x^n f(x)$ for some integer n, then

$$\tilde{g}(k) = (-ik)^n \tilde{f}(k) \ . \tag{4.19}$$

Proof:

$$\tilde{g}(k) = \int_{-\infty}^{\infty} dx g(x) e^{ikx} = \int_{-\infty}^{\infty} dx \frac{d^n}{dx^n} f(x) e^{ikx} \tag{4.20}$$

Now we can integrate by parts n-times, assuming $f(x \to \pm\infty) = 0$, to get

$$\tilde{g}(k) = \int_{-\infty}^{\infty} dx f(x) \left(-\frac{d}{dx}\right)^n e^{ikx} = (-ikx)^n \int_{-\infty}^{\infty} dx f(x) e^{ikx} = (-ik)^n \tilde{f}(k) \tag{4.21}$$

This observation is useful in both directions, as something you are trying to inverse transforms $\tilde{g}(k) = k^2 \tilde{f}(k)$ where $\tilde{f}(k)$ in the fourier transform of some function you know $f(x)$. You can find $g(x)$ by realizing

$$g(x) = -\frac{d^2}{dx^2} f(x) \ . \tag{4.22}$$

Part of what makes the Fourier transform useful analytically (as opposed to numerically, which we will discuss later), it that (a) we can evaluate a number of cases easily and (b) it has some useful general properties that simplify calculations.

Let's start with the simplest possible case: $f(t) = \cos\omega_0 t$. I am using t here to evoke the idea that this is time and ω_0 is a single frequency. Think of someone holding a single note on the piano. Now the fourier transform is

$$\begin{aligned} \tilde{f}(\omega) &= \int_{-\infty}^{\infty} dt e^{i\omega t} \cos\omega_0 t \\ &= \int_{-\infty}^{\infty} dt e^{i\omega t} \frac{1}{2}[e^{i\omega_0 t} + e^{-i\omega_0 t}] \\ &= \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)] \end{aligned} \tag{4.23}$$

So we see that the fourier transform of a pure wave picks out the the specific frequency as a δ-function. We can also use use the translation property to figure that if

$$\begin{aligned} f(t) &= \sin\omega_0 t = \cos\omega_0(t - \pi/(2\omega_0)) \\ \to \tilde{f}(\omega) &= e^{i\omega\pi/(2\omega_0)} \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)] \\ &= \pi[e^{-i\pi/2}\delta(\omega + \omega_0) + e^{i\pi/2}\delta(\omega - \omega_0)] \end{aligned} \tag{4.24}$$

Note two things here: (a) if I have $f(x)\delta(x)$ I can just replace it with $f(0)\delta(x)$ and (2) the relationship between fourier transform of a sine and cosine is just a constant phase $e^{i\pi/2}$.

Now let's try the more non-trivial case of an exponential

$$\begin{aligned} f(x) = e^{-b|x|} \to \tilde{f}(k) &= \int_{-\infty}^{\infty} dx e^{ikx} e^{-b|x|} \\ &= \int_0^{\infty} dk [e^{-(b-ik)x} + e^{-(b+ik)x}] \\ &= -\frac{1}{b-ik}[e^{-(b-ik)x}]|_0^{\infty} - \frac{1}{b+ik}[e^{-(b+ik)x}]|_0^{\infty} \\ &= \frac{1}{b-ik} + \frac{1}{b+ik} \\ &= \frac{2b}{b^2+k^2} \end{aligned} \tag{4.25}$$

We can similarly extend this to $x^n e^{-b|x|}$.

A very important example is the case of $f(x) = e^{-x^2/2\sigma^2}$. This is a Gaussian of variance σ. We will sketch the caculation now but we will revisit this when we discuss complex analysis:

$$\begin{aligned} f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} \to \tilde{f}(k) &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} dx e^{ikx} e^{-x^2/2\sigma^2} \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-k^2\sigma^2/2} \int_{-\infty}^{\infty} dx e^{-(x-ik\sigma^2)^2/2\sigma^2} \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-k^2\sigma^2/2} \int_{-\infty-ik\sigma}^{\infty-ik\sigma} dy e^{-y^2/2\sigma^2} \\ &= e^{-k^2\sigma^2/2} \end{aligned} \tag{4.26}$$

Remember this one! We will come back to talk about meaning of this formula for physics (e.g. the uncertainty principle!), but the fact that the Fourier transform of a Gaussian is a gaussian with $\sigma. \to 1/\sigma$ is something that something as important to remember as $F = ma$.

I should add an important remark here: I cheated you when doing this integral. I changed variables from x to $y = x - i\omega$ in the integration. This is perfectly fine, but now I am doing an integral in the complex domain, not just from $-\infty \to \infty$. The answer I quoted is correct, but we will come back to talk about these kinds of integrals because it is a much more delicate procedure than I let on and you will get into trouble in many other cases if you aren't careful about complex integrals.

4.3 Multi-Dimensional Fourier Transforms

To study more complicated examples, we are going to want to fourier transform in more than one coordinate at a time. A common situation is that we will have a function $\phi(t, \vec{x})$ where $\vec{x} = (x, y, z)^T$ is a 2/3 dimensional vector. To solve these types of problems we may fourier transform all of the coordinates to write

$$\tilde{\phi}(\omega, \vec{k}) = \int dt e^{i\omega t} \int dx e^{ik_x x} \int dy e^{ik_y y} \int dz e^{ik_z z} \phi(t, x, y, z) \tag{4.27}$$

$$\equiv \int dt \int d^3\vec{x} e^{i\omega t} e^{i\vec{x}\cdot\vec{k}} \phi(t, \vec{x}) \tag{4.28}$$

where $\vec{k} = (k_x, k_y, k_z)^T$. Given $\tilde{\phi}(\omega, \vec{k})$, we get $\phi(t, \vec{x})$ by inverting all of the transforms which we will write as

$$\begin{aligned} \phi(t,x,y,z) &= \int \frac{d\omega}{(2\pi)} e^{-i\omega t} \int \frac{dk_x}{(2\pi)} e^{-ik_x x} \int \frac{dk_y}{(2\pi)} e^{-ik_y y} \int \frac{dk_z}{(2\pi)} e^{-ik_z z} \tilde{\phi}(\omega, \vec{k}) \qquad (4.29) \\ &\equiv \int \frac{dt}{(2\pi)} \int \frac{d^3\vec{x}}{(2\pi)^3} e^{-i\omega t} e^{-i\vec{x}\cdot\vec{k}} \tilde{\phi}(\omega, \vec{k}) \qquad (4.30) \end{aligned}$$

There is nothing particularly special about doing higher dimensional fourier transforms, it is just repeating the operation on the multi-variables that define the function $\phi(t, \vec{x})$. In some situations, it may be helpful to only fourier transform some but not all of these variables, e.g. $\tilde{\phi}(t, \vec{k})$ we only fourier transform the space and not the time dependence.

4.4 Back to Physics

At the beginning of the section, we discussed how the wave equation for waves in the real world have speeds that depend on frequency. In particular, we said the the fourier transform is nothing more than breaking light into colors. We can see this explicitly if we apply the Fourier transform to the wave equation,

$$\frac{\partial^2 \phi(x,t)}{\partial t^2} - c_s^2 \frac{\partial^2 \phi(x,t)}{\partial x^2} = 0 \ . \qquad (4.31)$$

Specifically, we can represent this function in terms of its fourier transform in space and time

$$\phi(t,x) = \int_{-\infty}^{\infty} \frac{d\omega}{(2\pi)} e^{-i\omega t} \int_{-\infty}^{\infty} \frac{dk}{(2\pi)} e^{-ikx} \tilde{\phi}(\omega, k) \ . \qquad (4.32)$$

If we plug this into the equation we get

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2 \frac{\partial^2}{\partial x^2} \right) \int_{-\infty}^{\infty} \frac{d\omega dk}{(2\pi)^2} e^{-i\omega t - ikx} \tilde{\phi}(\omega, k) = 0 \qquad (4.33)$$

$$\int_{-\infty}^{\infty} \frac{d\omega dk}{(2\pi)^2} \left((-i\omega)^2 - c_s^2 (-ik)^2 \right) e^{-i\omega t - ikx} \tilde{\phi}(\omega, k) = 0 \qquad (4.34)$$

Now if we integrate both sides of this equation by

$$\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx e^{i\omega' t} e^{ik' x} [\text{LHS}] = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx e^{i\omega' t} e^{ik' x} [\text{RHS}] \qquad (4.35)$$

Integrating over x and t given $(2\pi)\delta(k - k')$ and $(2\pi)\delta(\omega - \omega')$ respectively. We can then integrate over k and ω to find

$$(-\omega'^2 + c_s^2 k'^2) \tilde{\phi}(\omega, k) = 0 \qquad (4.36)$$

This means that $\omega' = \pm c_s k$ and

$$\tilde{\phi}(\omega, k) = \tilde{\phi}_1(\omega)(2\pi)\delta(\omega - c_s k) + \tilde{\phi}_2(\omega)(2\pi)\delta(\omega + c_s k) \qquad (4.37)$$

Suppose we have light coming in from one direction so that $\phi_2 = 0$. The waveform we observe is completely determined by is colors

$$\phi(x,t) = \int \frac{d\omega}{2\pi} e^{i\omega(t + k/c_s)} \tilde{\phi}_1(\omega) \qquad (4.38)$$

Notice $\tilde{\phi}_1(\omega)$ takes complex values as needed to get the phases of the waves. In contrast, the amplitude of each color $|\tilde{\phi}_+(\omega)|$ is what you would see as the spectrum of light is missing of the information about the waveform.

Now we remember that we had this interesting case of light travelling in a medium where $\omega^2 = c_s^2(\omega)k^2$, or that the speed depends on the color of light. Now we are in a position to understand what this would look like in the differential equation. Specifically, we notice that we must get a product between $c_s(\omega)$ and $\tilde{\phi}(\omega, k)$ in fourier space. Now we recall that multiplication in fourier space is convolution in position space. Given that our $c_s^2(\omega)$ must be related to some $c_s^2(t)$ (inverse fourier transform), our "wave equation" takes the form

$$\frac{\partial^2\phi(x,t)}{\partial t^2} - \int dt'[c_s^2(t-t')\frac{\partial^2\phi(x,t')}{\partial x^2}] = 0 \tag{4.39}$$

In turns out it isn't a differential equation anymore but is an integro-differential equation. I won't say much about such equations other than to point out that we don't discuss things this way for a reason: it is not very useful. This, in essence, is why we think about waves as oscillations: waves are a much more complicated phenomena that the wave equation suggests but thinking about it as a bunch of oscillations is the most useful way to organize this breadth of phenomena.

The simplicity of physics in fourier space is a far more general phenomena that just waves with non-trivial dispersion. We also can see other physical phenomena are simple consequences of the fourier transform

- **Energy and Momentum Conservation:** You may have noticed that when we write many equations of motion in fourier space, the equations give some ω or k but they don't change over time or space. In some sense, you might say that ω and k are "conserved". In fact, this is not an accident: this is literally energy and momentum conservation in action.

  At our level of discussion, we noticed that if we have a linear equation that has no explicit dependence on x or t, then we can write it as an algebriac equation in terms of ω and k. In addition, each allowed pair of ω and k will evolve in time without changing. Furthermore, you can add all these solutions together and they will evolve independently.

  In quantum mechanics, the energy of a specific state is just $E = \omega\hbar$ and the momentum of the state is just $p = k\hbar$. Therefore, in a very literal sense, the fourier transform works because (when) energy and momentum are conserved.

- **The uncertainty principle:** The uncertainty principle follows from the statement that the Fourier transform of a Gaussian of width σ is another gaussian of width $1/\sigma$. In quantum mechanics, the position and momentum representation are related by the fourier transform so that $k \equiv p/\hbar$ if $f(x)$ is wavefunction for the position. Now, the probability of finding a particle at position x is $|f(x)|^2$ and momentum p is $|f(p)|^2$. As a result, if the wavefuncton has some width σ so that the fourier transform in k as width $1/\sigma$ then we have

  $$\sigma_x = \frac{\sigma}{\sqrt{2}} \qquad \sigma_p = \frac{\hbar}{\sqrt{2}\sigma} \qquad \rightarrow \qquad \sigma_x\sigma_p = \frac{\hbar}{2} \tag{4.40}$$

  which is the essence of the uncertainty principle.

- **Sidebands:** If you have used a radio, you might notice that you signal is present not just in the carrier frequency about also the frequencies nearby. The easiest example is AM radio, where you have a carrier frequency ω_0 and some message $g(t)$ so that the signal is

$$f(t) = g(t)\cos\omega_0 t \tag{4.41}$$

Here we imagine that $g(t)$ chances much more slowly than one period of oscillation $2\pi/\omega_0$.Now you tune your radio to a frequency ω_0 to pick up the signal. But if we Fourier transform we can see that actual frequencies that are carried are

$$\begin{aligned}\tilde{f}(\omega) &= \int dt g(t) e^{i\omega t} e^{i\omega_0 t} \\ &= .\int \frac{d\omega'}{(2\pi)} dt g(\omega') e^{-i\omega' t} e^{i\omega t} e^{-i\omega_0 t} \\ &= \tilde{g}(\omega - \omega_0)\end{aligned} \tag{4.42}$$

So the fact that $g(t)$ is slowly varied means that $g(\omega')$ is peaked near zero. However, because $g(t)$ is some non-trivial function (i.e. not a constant or a polynomial), $g(\omega' \neq 0) \neq 0$ and therefore the signal that was originally a pure frequency $\omega = \omega_0$ will bleed into adjacent frequencies. For example, if you signal was itself a pure frequency ω_1 so that $g(\omega') = \delta(\omega' - \omega_1)$ then our signal would be $\tilde{f}(\omega) = \delta(\omega - \omega_0 - \omega_1)$.

5 Introduction to Green's Functions

Dubin Chapter 2.4

5.1 Sources and Green's Functions

Green's functions are a very powerful tool in solving differential equations and all kinds of physics problems. They can also seem a bit intimidating. But believe it or not you are all already familiar with a least a few examples fo Green's functions. Physically, the Green's function is a object that tells you what happens when you disturb a system in some way. In particular, if I do something in one place, that information will propagate and the Green's functions tells us how this happens.

To give you an example you all should know, let us recall Gauss's laws from electrostatics. It is written as differential equation as

$$(\partial_x^2 + \partial_y^2 + \partial_z^2)\Phi(x) = \frac{1}{4\pi\epsilon_0}\rho(x) \tag{5.1}$$

where Φ is the electro-static potential and $\rho(x)$ is the density of electric charges. You have also all learned that if you have a single change q at location $x = 0$, then the potential away from that charge is

$$\Phi(x) = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{x^2+y^2+z^2}} \ . \tag{5.2}$$

You then lean that since electro-statics is linear, you can get the answer for any change distribution just by adding up all of the individual changes

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\rho(x') \tag{5.3}$$

You should have reasonably good intuition for this formula: each electric charge generates an electric field and we are simply adding up all the contribution to the electric field from all of those changes.

On the other hand, this formula is one we will see over and over again in the context of Green's functions and beyond. If I defined the function

$$G(x) = \frac{1}{\sqrt{x^2+y^2+z^2}} \tag{5.4}$$

then our formula is

$$\Phi(x) = \int d^3x' G(\vec{x}-\vec{x}')\frac{1}{4\pi\epsilon_0}\rho(x') \tag{5.5}$$

where $G(\vec{x}-\vec{x}')$ is the Green's function defined by Gauss's law. In this course will built up some tools for how to actually find the Green's function given a differential equation. But, intuitively this is all that is happening. The Green's function is just a generalization of the rule for how adding an electric charge influences the electric potential: I find the answer for what happens for a single point-like "charge" and then I add them all up.

5.2 Sources and the Harmonic Oscillator

Now let us go back to a simpler case, the harmonic oscillator with a general driving force $h(t)$:

$$\ddot{x}(t) + \gamma\dot{x}(t) + \omega_0^2 x(t) = h(t) \tag{5.6}$$

Now we fourier transform both sides of this equation

$$\int dt e^{i\omega t}\left[\ddot{x}(t) + \gamma\dot{x}(t) + \omega_0^2 x(t) \right] = h(t)\right] \tag{5.7}$$

$$(\omega_0^2 - \omega^2 - i\gamma\omega)\tilde{x}(\omega) = \tilde{h}(\omega) \tag{5.8}$$

Here we used integration by parts to get, for example, $\int dt e^{i\omega t}\ddot{x}(t) = -\int dt(i\omega)e^{i\omega t}\dot{x}(t) = -\omega^2\int dt e^{i\omega t}x(t)$. Here we are implicitly using one of two assumptions

- $x(t = \pm\infty) = 0$: This means when we integrate by parts, the terms like $e^{i\omega t}x(t)|_{-\infty}^{\infty} = 0$ because the function itself is zero.
- $x(t = \pm\infty) \neq 0$ but $x(t \to \infty) = x(t \to -\infty)$ is periodic: If $x(t\pm\infty) \neq 0$, this integral is not well-defined as the integral formally diverges. To define the integral in these cases, we recall that the fourier transform was just the limit of the fourier series and we must (secretely) still be imposing periodic boundary conditions. Therefore, we still have $e^{i\omega t}x(t)|_{-\infty}^{\infty} = 0$ because of the implicit periodic assumption.

If the integration by parts makes you uncomfortable for this reason, we could derive the same equation by plugging in the definition of $x(t)$ in terms of $\tilde{x}(\omega)$:

$$\left(\frac{d^2}{dt^2} + \gamma\frac{d}{dt} + \omega_0^2\right)\int \frac{d\omega}{2\pi}e^{-i\omega t}\tilde{x}(\omega) = \int \frac{d\omega}{2\pi}e^{-i\omega t}\tilde{h}(\omega) \tag{5.9}$$

$$\int \frac{d\omega}{2\pi}e^{-i\omega t}(\omega_0^2 - \omega^2 - i\gamma\omega)\tilde{x}(\omega) = \int \frac{d\omega}{2\pi}e^{-i\omega t}\tilde{h}(\omega) \tag{5.10}$$

Since we can't cancel the time dependence between different frequencies at all times, the only way to solve this equation is if it is true for every ω:

$$(\omega_0^2 - \omega^2 - i\gamma\omega)\tilde{x}(\omega) = \tilde{h}(\omega) \tag{5.11}$$

Both ways of arriving at this equation are correct, so feel free to use whichever derivation is easier to understand for you.

Solving for the influence of the source $h(t)$ might not have been obvious, but now in fourier space it is just algebra

$$\tilde{x}(\omega) = \frac{1}{(\omega_0^2 - \omega^2 - i\gamma\omega)}\tilde{h}(\omega) \tag{5.12}$$

We are now going to define the Green's function by its fourier transform

$$\tilde{G}(\omega) = \frac{1}{(\omega_0^2 - \omega^2 - i\gamma\omega)} \tag{5.13}$$

so that in fourier space

$$\tilde{x}(\omega) = \tilde{G}(\omega)\tilde{h}(\omega) \ . \tag{5.14}$$

Using the convolution property of the fourier transform, we can see that this is exactly what we could have expeted

$$\begin{aligned} G(t) &= \int \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{(\omega_0^2 - \omega^2 - i\gamma\omega)} \\ x(t) &= \int dt' G(t-t')h(t') \ . \end{aligned} \tag{5.15}$$

This is the same property we found before: when the fourier transform turns a differential equation into an algebraic equation, the problem of finding the behavior form are arbitrary source similarly becomes algebraic in fourier space.

5.3 Matching to Previous Solution

Now we consider the situation where we drive the system with an external form $F_{\text{ext}} = ma_0 \cos(\omega' t)$. *Let us recall how we solved this case:*

$$\ddot{x}(t) + \gamma\dot{x}(t) + \omega_0^2 x(t) = a_0 \cos(\omega' t) : \tag{5.16}$$

Now we should be careful about how to make this a complex equation. Note that if we are solving

$$[\partial_t^2 + \gamma\partial + \omega_0^2][x_0 + \text{c.c}] = \frac{1}{2}a_0 e^{i\omega' t} + \text{c.c.} \tag{5.17}$$

then if we can solve

$$[\partial_t^2 + \gamma\partial + \omega_0^2]x_0 = \frac{1}{2}a_0 e^{i\omega' t} \tag{5.18}$$

when we automatically solve the equation. Let's try our ansatz again:

$$Ae^{\rho t}[\rho^2 + \rho\gamma + \omega_0^2\rho] = \frac{1}{2}a_0 e^{i\omega' t} \tag{5.19}$$

First of all, we must have $\rho = i\omega'$ if we have any hope of solving this equation at every t. Solving for A we get

$$A = \frac{1}{2}\frac{a_0}{-\omega'^2 + i\gamma\omega + \omega_0^2} = \frac{a_0}{2}\frac{(\omega_0^2 - \omega'^2) - i\omega'\gamma}{(\omega_0^2 - \omega'^2)^2 + (\omega'\gamma)^2} \tag{5.20}$$

and therefore

$$x(t) = x_{\text{homogenous}} + \frac{a_0}{(\omega_0^2 - \omega'^2)^2 + (\omega'\gamma)^2}\left[(\omega_0^2 - \omega'^2)\cos\omega' t + \omega\gamma \sin\omega' t\right] \tag{5.21}$$

We see the usual intuition that if we drive the system near the natural frequency $\omega' = \omega_0$, the we get a large amplitude oscillation in response. If we assume $\gamma\omega > |\omega_0^2 - \omega'^2|$ then

$$x(t) \approx \frac{a_0}{\omega_0\gamma}\sin\omega_0 t \tag{5.22}$$

which is parametrically larger than the force itself when $\omega_0\gamma \ll a_0$. I.e. when the damping is very small, we can fine tune the frequency so that a small external force produces very large response.

Notice that this way of solving secretly used the Green's function in the form

$$G(\omega \to \omega') = \frac{1}{(\omega_0^2 - \omega'^2 + i\gamma\omega')} = \frac{(\omega_0^2 - \omega'^2) - i\omega'\gamma}{(\omega_0^2 - \omega'^2)^2 + (\omega'\gamma)^2} \tag{5.23}$$

In fact, all we have done is take the case where $h(\omega) = a_0\pi[\delta(\omega - \omega') + \delta(\omega + \omega')]$ so that

$$\begin{aligned} \int \frac{d\omega}{2\pi} G(\omega)h(\omega)e^{i\omega t} &= \frac{a_0}{2}\frac{(\omega_0^2 - \omega'^2)(e^{i\omega' t} + e^{-i\omega' t}) + i\omega'(e^{i\omega' t} + e^{-i\omega' t})}{(\omega_0^2 - \omega'^2)^2 + (\omega'\gamma)^2} & (5.24) \\ &= \frac{a_0}{(\omega_0^2 - \omega'^2)^2 + (\omega'\gamma)^2}\left[(\omega_0^2 - \omega'^2)\cos\omega' t + \omega\gamma\sin\omega' t\right] & (5.25) \end{aligned}$$

When we put in a pure frequency, the Green's function is just implementing the inhomogenous solution in the way you would have just by guessing. Now we get the general inhomogeous solution without just by adding up all the possible pure frequencies.

But what happened to the homogenous solutions. When we solved this equation we just used algebra in the most obvious way. However, I made a seemingly innocuous decision to divide by $(\omega_0^2 - \omega^2 - i\gamma\omega)$ but the homogeneous solution are precisely those where this expression is zero. When $\gamma = 0$, it is clear what we should do:

$$x(\omega) = \kappa\delta(\omega - \omega_0) + \kappa^*\delta(\omega + \omega_0) + G(\omega)h(\omega)\ . \tag{5.26}$$

But what happens when $\gamma \neq 0$ and the only solutions to $(\omega_0^2 - \omega^2 - i\gamma\omega) = 0$ are for ω are complex. But ω is a real number and therefore there are no additional solutions. To see the problem, let's go back to real space and remember our homogenous solution was

$$x(t) = e^{-\gamma t/2}[\kappa e^{i\omega t} + \text{c.c.}] \tag{5.27}$$

We are trying to define this function on the domain $t \in [-\infty, \infty]$ but clearly $x(t) \to \infty$ as $t \to -\infty$. As a result, the only well defined solution is with $\kappa = 0$. The Fourier transform was indeed telling us something important!

5.4 Sources and the Wave Equation

The wave equation captures both some of the complexity of higher dimensional examples we will encounter later in the course, but it is also situation you likely have good physical intuition for. In particular, if you have ever thrown a rock into a still pond, you probably have a sense of what happens when you source a wave at a specific place at a specific time. Let's see how that intuition manifests itself in mathematical form.

Like the harmonic oscillator, we are going to take an equation we have already solve and add an arbitrary "source" on the RHS of the equation:

$$\frac{\partial^2\phi(x,t)}{\partial t^2} - c_s^2\frac{\partial^2\phi(x,t)}{\partial x^2} = h(t,x) \tag{5.28}$$

Now, as before, we are going to fourier transform this equation in two variables, so that

$$\phi(\omega, k) = \int dtdx\phi(x,t)e^{i\omega t + ikx} \tag{5.29}$$

and therefore

$$\begin{aligned} & -(\omega^2 - c_s^2 k^2)\phi(\omega,k) = h(\omega,k) \\ \rightarrow \quad & \phi(\omega,k) = -\frac{h(\omega,k)}{(\omega^2 - c_s^2 k^2)} + \phi_1(k)\delta(\omega - c_s k) + \phi_2(k)\delta(\omega + c_s k) \end{aligned} \tag{5.30}$$

Here I have added the homogenous conditions back in terms of solutions to $(\omega^2 - c_s^2 k^2) = 0$ by using δ-functions.

Just like the harmonic oscillator, we can fourier transform the Green's function back to space and time,

$$G(t,x) = \int \frac{d\omega dk}{(2\pi)^2} \frac{-1}{(\omega^2 - c_s^2 k^2)} e^{-i\omega t - ikx} \tag{5.31}$$

so that we have

$$\phi(x,t) = \int dt' dx' G(t-t', x-x') h(t',x') + \phi_1(x + c_s t) + \phi_2(x - c_s t) \tag{5.32}$$

This is all very dry because I haven't told you what $G(t,x)$ is, so let's figure it out. To make the calculation easier, I also what to specify the source is just a constant frequency in time, ω_0 but a δ-function in space:

$$h(t,x) \propto \cos\omega_0 t \times \delta(x) \rightarrow h(\omega,k) = (2\pi)[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] \tag{5.33}$$

We can now compute the response

$$\begin{aligned} \phi(x,t) &= \int \frac{d\omega dk}{(2\pi)^2} e^{i\omega t + ikx} G(\omega,k) h(\omega,k) \\ &= e^{i\omega_0 t} \int \frac{dk}{(2\pi)} e^{ikx} \frac{1}{\omega_0^2 - c_s^2 k^2} + \{\omega_0 \rightarrow -\omega_0\} \\ &= \frac{i}{2\omega_0}\left[e^{-i\omega_0 t - i\omega_0 |x|/c_s|} - e^{+i\omega_0 t + i\omega_0 |x|/c_s|}\right] \\ &= \frac{1}{\omega_0}\sin(\omega_0 t - \omega_0 |x|/c_s)) \end{aligned} \tag{5.34}$$

Understanding how to do this integral correctly will again lead us to complex integration.

We can look at the final answer and see something physically intuitive. We are driving the system with an oscillation at a pure frequency ω_0 at the location $x = 0$. If I stand at $x = 0$, this means that I find

$$\phi(x=0,t) = \frac{1}{\omega_0}\sin(\omega_0 t) \tag{5.35}$$

The phase and amplitude don't quite match $h(t,x)$ but that is to be expected (the response of ϕ need not match $h(t)$ exactly), but I am still seeing an oscillation with frequency ω_0.

Now consider what happens if I am standing at a position $x > 0$. I now see

$$\phi(x,t) = \frac{1}{\omega_0}\sin((\omega_0(t - x/c_s))\ . \tag{5.36}$$

which is the same oscillation I saw at $x = 0$ that I see at time t what happened at $x = 0$ at a time by $t - x/c_s$. In other words, what I see is delayed by a time x/c_s, which is the time it takes a wave to propagator from $x = 0$ to x (remember c_s is the speed of the wave).

In the end, this Green's function is telling you something you should have expected anyway: if I disturb the system at $x = 0$, what I see at $x > 0$ is just the wave sourced at $x = 0$ propagated at a speed c_s. In higher dimensions, this formula essentially the same, but include some additional powers of x^{-1} to account for the attenuation of the wave as it propagates (i.e. as the ring travels outwards, it must conserve energy as the radius gets larger and thus the amplitude goes down as x increases.)

6 Basics of Complex Analysis

Wikipedia:Complex Analysis
Youtube Khan Acadamy:Complex Functions

6.1 Why Complex Analysis?

The need for complex analysis can be traced to basic physical principles. To see this, let us recall that the evolution of a harmonic oscillator with a generic external force, $F_{\rm ext} = mh(t)$, is given by the equation

$$\ddot{x}(t) + \gamma\dot{x}(t) + \omega_0^2 x(t) = h(t) \tag{6.1}$$

In fourier space, we could solve this equation algebraically. So that

$$\tilde{x}(\omega) \quad = \quad \frac{1}{(\omega_0^2 - \omega^2 - i\gamma\omega)}\tilde{h}(\omega), \tag{6.2}$$

where the Green's function is

$$\tilde{G}(\omega) = \frac{1}{(\omega_0^2 - \omega^2 - i\gamma\omega)} \ . \tag{6.3}$$

In real space, we know this must be a convolution

$$x(t) = \int_{-\infty}^{\infty} dt' G(t-t')h(t') \ . \tag{6.4}$$

There is something bizarre about this formula, at first sight: the integral including $t' > t$ so that it, naively, looks like $h(t' > t)$ is affecting $x(t)$. Stop to think about this for a second - it is suggesting that the location of the particle at time t is affected by the force at some time later. This would violate our most basic notions of cause and effect, namely that the future does not affect the past.

The resolution to this puzzle is that the Green's function must vanish for $t' > t$, $G(t - t' < 0) = 0$, so that there is actually no contribution from $h(t' > t)$ to $x(t)$. This is an absolutely necessary property of $G(t - t')$. But this is hardly obvious from the fourier transform

$$G(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{(\omega_0^2 - \omega^2 - i\gamma\omega)} e^{-i\omega t} \ . \tag{6.5}$$

What somehow must be true is that

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{(\omega_0^2 - \omega^2 - i\gamma\omega)} e^{-i\omega t} \quad = \quad 0 \qquad t < 0 \tag{6.6}$$

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{(\omega_0^2 - \omega^2 - i\gamma\omega)} e^{-i\omega t} \quad \neq \quad 0 \qquad t \geq 0 \tag{6.7}$$

Without knowing complex analysis, this would seem to be some kind of miracle. For physics to work, we require some apparently magical property of integration to hold. The reason we are going to learn some complex analysis is that there is actually a very simple reason that the above integral behaves are required.

6.2 Complex Functions

Wikipedia: Holomorphic functions

We saw that Green's functions in Fourier space are fairly simple to write down but they are complex valued functions of $\omega \in \mathbb{R}$. In order to understand some of the special properties of the Green's function, it is useful to think of the function $G(\omega)$ as being a function of a complex variable $\omega \in \mathbb{C}$. Of course $G(\omega)$ itself is always complex valued but what we are doing is trying to understand the behavior of $G(\omega)$ for real values of ω by assigning it meaning when ω is complex. In other words, we will consider $G(z)$ where $z = x + iy$, even though we will eventually only worry about the situation where $y = 0$ and $x \leftrightarrow \omega$.
We will think of a complex function

$$f(z = x + iy) = u(x, y) + iv(x, y) \tag{6.8}$$

On the surface this is a trivial extension of $f(x) \to f(z)$. But if we now start demanding that it makes sense in terms of z, we will find some remarkable results.

Holomorphic Functions

A function is said to be holomorphic in some open set U if it has a well-defined derivative:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{6.9}$$

This sounds trivial, but remember that this only makes sense if it is $f'(z)$ is independent of how you take the limit. This is clear if we consider the following *non-holomorphic* function $g(z) = z + \bar{z}$ where I have defined $\bar{z} \equiv z^* = x - iy$. Of course, this is nothing more than the function $g(z) = 2\mathrm{Re}z = 2x$. However we can now think about taking a derivative at the point $z_0 = x_0 + iy_0$ two different ways. First, we could take the limit purely in the x-direction

$$f'_x(z_0 = x_0 + iy_0) = \lim_{x \to x_0} \frac{f(x + iy_0) - f(x_0 + iy_0)}{x - x_0} = 2 \ . \tag{6.10}$$

Alternatively, we could take the limit only in the y-direction

$$f'_y(z_0 = x_0 + iy_0) = \lim_{y \to 0} \frac{f(x_0 + iy) - f(x_0 + iy_0)}{i(y - y_0)} = 0 \tag{6.11}$$

These two limits clearly do not agree and therefore we do not have an ambiguous notation of $f'(z_0)$. Instead, a function if only *holomorphic* we can the same answer no matter how we approach the point z_0

To understand the requirements for a well-defined derivative, let us break up our any derivative into a limit along the real axis, $z = h + z_0$, or imaginary axis, $z = ih + z_0$, for real h. Requiring that the function in holomorphics means the derivative is independent of this choice. Concretely, we have

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \quad = \quad \lim_{h \to 0} \frac{f(z_0 + ih) - f(z_0)}{ih}$$

The LHS is just a derivative with respect to the real part of z which is x and the RHS is the imaginary part, y, which gives

$$\frac{\partial}{\partial x}f(x,y) = \frac{1}{i}\frac{\partial}{\partial y}f(x,y) \qquad \frac{\partial}{\partial x}f(x,y) - \frac{1}{i}\frac{d}{dy}f(x,y) = 0 \; . \tag{6.12}$$

The combination of derivatives on the left hand side can be written more elegantly by noticing

$$\frac{\partial}{\partial \bar{z}}f(z) = \frac{\partial f(x,y)}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f(x,y)}{dy}\frac{\partial y}{\partial \bar{z}} \tag{6.13}$$

$$= \frac{df(x,y)}{dx} - \frac{1}{i}\frac{df(x,y)}{dy} \tag{6.14}$$

where we used $\frac{\partial}{\partial y}\bar{z} = -i$ and $\frac{\partial}{\partial x}\bar{z} = 1$. As a result, a function is holomorphic when

$$\frac{\partial}{\partial \bar{z}}f(z) = 0 \tag{6.15}$$

In other words, we have to write it purely in terms of z and not $\bar{z}$. Now if we take a derivative with respect to z we have

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}f(z) = 0 \rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f(x,y) = 0 \tag{6.16}$$

In other words, $f(x,y)$ and hence u and v solve the Laplace equation!

Aside: Splitting $f = u + iv$ this means

$$i\frac{\partial}{\partial x}(u(x,y) + iv(x,y)) = \frac{\partial}{\partial y}u(x,y) + i\frac{\partial}{\partial y}v(x,y) \tag{6.17}$$

or

$$\frac{\partial}{\partial x}u = \frac{\partial}{\partial y}v \qquad \frac{\partial}{\partial y}u = -\frac{\partial}{\partial x}v \tag{6.18}$$

These equations are knows as the Cauchy-Riemann equations.

Analytic Functions

A function is said to be analytic in an open set U if is has convergence Taylor series:

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d^n}{dz^n}f(z)|_{z=z_0}(z - z_0)^n \tag{6.19}$$

for any point $z_0 \in U$. Technically speaking, the use of analytic is also applicable to real function as well.

A function f can be shown to be analytic *if and only if* it is holomorphic. As a result, the use of holomorphic and analytic are often used interchangeably. However, analytic has a precise meaning even when f is a real valued function or a function of a single real variable. Holomorphic is only a statement about complex function.

6.3 Branch Points, Cuts and Poles

Wikipedia: Branch Cuts
Wikipedia: Zeros and Poles

Since holomorphic / analytic functions must obey the Laplace equation, we can use our extensive knowledge of electrostatics to gain intuition for these functions. In particular, we know that there are no interesting solutions to the Laplace equations without sources. But what happened to the richness of the space of functions we know and love on the real line? The answer is that (almost) all of the interesting structure of complex functions is encoded in now it fails to be analytic in specific regions. If we think of this points (or regions) like charges, then just like electrostatics, complex functions will be uniquely determined by these non-analytic points.

There are several kinds of non-analytic behavior we need to worry about.

- **Poles:** The most basic type of non-analytic term are the *poles*. These are easy to describe: a function $f(z)$ has a pole of order n (where n is a positive integer) at $z = z_0$ if, in the vicinity of z_0 is can be described as

$$f(z) \approx \frac{1}{(z - z_0)^n} \sum_{m=0}^{\infty} c_m (z - z_0)^m \tag{6.20}$$

  Said differently, if $f(z)$ has a pole of order n at $z = z_0$ then $g(z) = f(z)(z - z_0)^n$ is a analytic function in some open set that includes z_0. Poles of order $n = 1$ are called *simple poles*.

  Two quick comments: (1) It is essential that n is an integer. We will see what happens when n is not an integer next. (2) It is also important to note that not every divergence at a point is a pole. For example, there is also something called an essential singularity, which is basically anything that is not a pole. The canonical example is

$$f(z) = e^{1/z} = \sum_{m=1}^{\infty} \frac{1}{m!} \frac{1}{z^m} \tag{6.21}$$

  So we see that, using the series expansion of the exponential, that there is no finite n we can choose such that $f(z)z^n$ is an analytic function at $z = 0$ (we won't really talk about these much in this course).

- **Branch Points:** A branch point is fairly easy to explain: a point z_0 is branch point of the a function $f(z)$ is $f(z)$ is discontinuous when we going around the point z_0, i.e. $z = z_0 + \epsilon e^{i\theta}$ with $\theta \in [0, 2\pi]$, for arbitrarily small ϵ.

  It is easiest to explain with the example $f(z) = z^{1/2}$. This function has a branch point at $z = 0$. If we go around this point we have

$$f(z = \epsilon e^{i\theta}) = \epsilon^{1/2} e^{i\theta/2} \tag{6.22}$$

  So that $\theta = 0$ we have $f(z = \epsilon_+) = 1$ but then we we take $\theta = 2\pi$ we have $f(z = \epsilon_-) = e^{i\pi} = -1$. Therefore, we see that the naive definition of the square root is discontinuous.

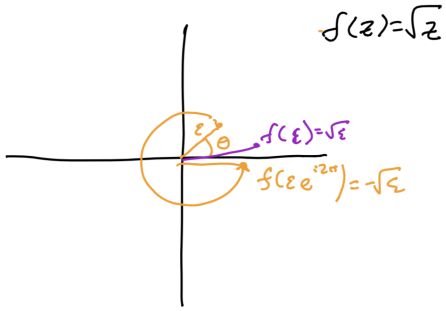

Figure 5: Branch point for $f(z) = \sqrt{z}$ at $z = 0$.

From this argument, we can see the z^α will have a branch point at $z = 0$ for any non-integer α. I.e. for $f(z) = z^\alpha$

$$f(\epsilon e^{i2\pi}) = \epsilon^\alpha e^{i2\pi\alpha} \neq \epsilon^\alpha = f(\epsilon) \tag{6.23}$$

for $\alpha \neq 0, \pm 1, \pm 2, \ldots$. We see the difference between poles and branch points, the poles are divergent at z_0 but are well behaved everywhere else. Not so for the branch points (e.g. non-integer powers). (See Figure 5)

Remark: Notice that whether or not $z = 0$ is a branch point has nothing to do with whether $f(z)$ diverges at $z = 0$ or its derivatives diverge. E.g. $f(z) = z^{-2}$ is not a branch point but everything diverges there. Similarly $f(z) = z^{11/2}$ is a branch point, but the function and its first derivative, $f'(z) = \frac{11}{2} z^{9/2}$ approach zero.

- **Branch Cut:** What is happening here is that $f(z)$ is actually a multi-valued function. However, it is useful to define $f(z)$ so that it is single valued over the complex plane, at the cost of introducing *branch cuts*. A branch cut is a curve (or curves) we define in $\mathbb{C}$ that we define such that our function is single valued and is only discontinuous across the curve. Where we put this curve is part of defining which of the many possible values we are choosing for $f(z)$.

  Consider two options for $f(z) = z^{1/2}$. Option 1: branch cut from $z = -\infty$ to $z = 0$ so that we define $z = \rho e^{i\theta}$ with $\theta \in (-\pi, \pi]$ (See Figure 6). We can see now that the function is continous on the positive real axis $f(z = e^{i\delta \to 0} = 1 = f(z = e^{-i\delta \to 0} = 1$. But we see the discontinuity across the branch cut:

$$f(z = -1 + i\epsilon \to -1) = e^{i\pi/2} = i \neq f(z = -1 - i\epsilon \to -1) = e^{-i\pi/2} = -i \tag{6.24}$$

  Option 2: the branch cut is along the positive real axis so that $z = \rho e^{i\theta}$ with $\theta \in [0, 2\pi)$. As before, this lead to a discontinuity on the real line.

  The later choice is not commonly used because it is nice to keep the funny behavior away from the positive real axis where we think functions like $\sqrt{x}$ are well behaved.

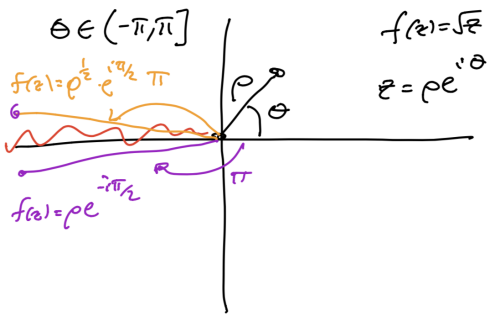

Figure 6: Branch cut for $f(z) = \sqrt{z}$ at $z = 0$.

The next important example is the complex logarithm. The need for a branch cut in this case is pretty clear from the familiar fact that $e^{i\theta} = e^{i\theta+i2\pi n}$. As a result, when we define

$$\log e^{i\theta} = \log e^{i\theta+i2\pi n} = i\theta + i2\pi n \ . \tag{6.25}$$

Clearly my definition of log is going to require that I pick what range of θ so that the log is single valued.

It is again, natural to pick the branch cut to be on the negative real axis. This means that the phase lies between $(-\pi, \pi]$ so the discontinuity along the negative axis by taking the log of $e^{\pm i\pi} = -1$.

The imaginary part of the complex logarithm is also called that *arguement*. So, if you are given a complex number in polar-coordinates, $z = \rho e^{i\theta}$ then

$$\log z = \log |z| + i \arg z = \log \rho + i\theta \pmod{2\pi} \ . \tag{6.26}$$

6.4 Examples

The most non-intuitive part of complex analysis (in my opinion) is where to put branch cuts. There isn't a unique choice which is part of what makes it confusing. However, our goal is to define the function in a way that is maximally useful. This is sort of what the whole business of complex analysis is about: we had many options in taking $f(x) = \sqrt{x}$ and inventing a function $f(z)$ such that $f(z) = \sqrt{x}$ when $z = x + i0$. But if we replaced every x with z (and not $\bar{z}$) then we get a function that is holomorphic and analytic almost everywhere and this will be a useful property. Now we are defining where to make $f(z)$ non-analytic in order to make it single valued with preserving as many nice properties as possible.

The best way to understand this is just to try a bunch of examples. Let's consider $f(z) = \sqrt{z(z+1)}$. We can see it has two branch points, $z = 0$ and $z = -1$ as follows (See Figure 7). First, to see $z = 0$ is a branch point, take $z = \epsilon e^{i\theta}$ so that

$$f(\epsilon e^{i\theta}) = \sqrt{\epsilon e^{i\theta}}\sqrt{1 + \epsilon e^{i\theta}} \to \sqrt{\epsilon} e^{i\theta/2} \tag{6.27}$$

Let's do the same for $z = -1$ by considering $z = -1 + \epsilon e^{i\theta}$ to get

$$f(-1 + \epsilon e^{i\theta}) = \sqrt{-1}\sqrt{\epsilon} e^{i\theta/2} \tag{6.28}$$

In both cases, taking $\theta \in [0, 2\pi)$ we see that $f(\theta = 0) = -f(\theta = 2\pi)$. For $z = -1$, we also see that we need to pick a branch of $\sqrt{-1}$ but this doesn't depend on θ.

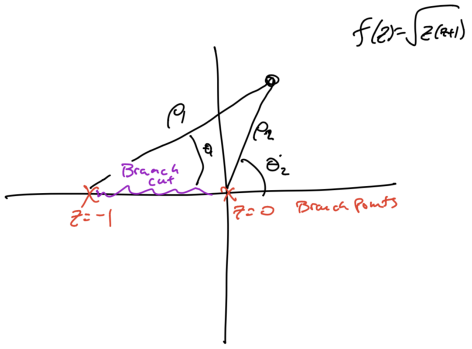

Figure 7: Branch cut for $f(z) = \sqrt{z(z+1)}$ at $z = 0$.

Now we want to pick some brach cuts that make $f(z)$ single valued at every point in the complex plan. A useful rule of thumb is that we should try to put the branch cut between the two branch points. This will not always be possible, but in this case it is. One way to see why if is possible not to have a brach cut that runs to $\pm\infty$ is that if you take $z = \rho e^{i\theta}$ with $\rho \gg 1$ then

$$f(z) \approx \sqrt{\rho^2 e^{2i\theta}} = \rho e^{i\theta} \tag{6.29}$$

so that the function is single valued if we go around both branch points. Noticed that this wouldn't work if we had some different fraction power of $z(z+1)$.

So how do we put the branch cut between the branch points? One way we do this is as follows: given a point z we define

$$z + 1 \equiv \rho_1 e^{i\theta_1} \tag{6.30}$$

where $\theta_1 \in (-\pi, \pi]$ and

$$z = \rho_2 e^{i\theta_2} \tag{6.31}$$

with $\theta_2 \in (-\pi, \pi]$. Now let us check there the function $f(z)$ has discontinuities. The general answer is

$$f(z) = \sqrt{\rho_1 \rho_2} e^{i(\theta_1+\theta_2)/2} \tag{6.32}$$

We will go to the real axis with $z > 0$ we have

$$f(z > 0) = \sqrt{\rho_1 \rho_2} e^{i\theta_1/2} e^{i\theta_2/2} \to \sqrt{\rho_1 \rho_2} \tag{6.33}$$

because both $\theta_2, \theta_1 \to 0$. We see that there is no discontinuity. For $z < -1$ we have

$$f(z < -1) = \sqrt{\rho_1 \rho_2} e^{i(\theta_1+\theta_2)/2} = -\sqrt{\rho_1 \rho_2} \tag{6.34}$$

It is not obvious this is continuous but we have $\theta_1 + \theta_2 = \pm 2\pi$ above and below the real line. However, $e^{\pm i\pi} = -1$ so it is actually continuous. It is only between the two points that we have $\theta_1 + \theta_2 = \pm\pi$ and

$$f(-1 < z < 0) = \sqrt{\rho_1 \rho_2} e^{i(\theta_1+\theta_2)/2} = \pm i\sqrt{\rho_1 \rho_2} \tag{6.35}$$

Now let's see how this works by calculating $f(z=i)$. First we determine

$$z+1 = i+1 = \rho_1 e^{i\theta_1} = \sqrt{2}e^{i\pi/4} \qquad z = i = \rho_2 e^{i\theta_2} = e^{i\pi/2} \tag{6.36}$$

so that

$$f(z=i) = \sqrt{\rho_1\rho_2}e^{i(\theta_1+\theta_2)/2} = 2^{1/4}e^{i3\pi/8} \ . \tag{6.37}$$

This is more or less what we would have thought to do, expect we just need to be careful how we pick θ in polar coordinates.

It might not be obvious at this point, but you might realize that when you put things in polar coordinates, you are just taking a bunch of logs. In fact, we could think of the entire class of non-integer powers as

$$z^\alpha = e^{i\alpha \log z} \tag{6.38}$$

This might as well be a definition of $\log z$. Since e^z is a nice analytic function everywhere, it means that the branch cut is just the same as the branch cut of $\log z$. Therefore, we can pick the branch cuts for $z^{1/3}, z^{1/4}$ or even z^π to lie along the negative real axis.

It is not always useful to use the branch cuts of the log, however. If we consider the case

$$h(z) = (z^2-1)^{1/2} \tag{6.39}$$

we can think of this two ways:

$$h(z) = (z-1)^{1/2}(z+1)^{1/2} \qquad \text{or} \qquad h(z) = e^{\frac{1}{2}\log(z^2-1)} \tag{6.40}$$

Recall that there is a branch of $\log w$ whenever w is real and negative. But this means we are on the branch cut for $z = iy$ for any y and $z = x$ when $|x| < 1$. So the branch cuts from the log look pretty ugly. On the other hand, I can solve this problem in the exact same way as $f(z) = \sqrt{z(z+1)}$ and just had a branch cut running between the two branch points $z = \pm 1$.

7 Complex Integration

Wikipedia:Contour Integration

In the previous section, we took a function $f(x)$ and asked how to define $f(z)$ if we invented a complex variable $z = x + iy$. From our point of view, this is a trick since we are usually interested in functions of a real variable. You can think of this as being similar to the problems of integrating $x^{2n}e^{-x^2/2}$ and noticing that it is a good idea to invent a parameter b and integrate $e^{-bx^2/2}$ and taking derivatives with respect to b. Although your answer is independent of b, it is a very useful invention to find the answer, even though you take $b = 1$ at the end.

We are now going to see the payoff for inventing an imaginary partner to our real variable. Like b, it is going to make a bunch of integrals much easier to evaluate. Unlike b, it is not as obvious from the beginning so we need a few theorems before it will make sense.

7.1 Smooth Arcs and Contours

We are going to be interested in integrals in the complex plane over a smooth arc or contour. A *smooth arc* is just a smooth line in the z-plane that we can write as a map from a real number t into z:

$$z_\gamma(t) = x(t) + iy(t) \tag{7.1}$$

Just think of this as a parametric plot in 2d dimensions. In order to be useful, we will require that $\dot{z}(t) = \dot{x}(t) + i\dot{y}(t) \neq 0$ for any $t \in [a, b]$, where $\dot{x} \equiv \frac{d}{dt}x(t)$ as before. We will call this curve smooth if z' is continuous and the map $t \to z$ is one-to-one (i.e. each t gives a unique value of z). We also can definite a *directed arc* by giving meaning the the order through which you go through the points z assuming you start at $t = a$ and go to $t = b$ (we want to to this because we want to distinguish integration $\int_0^1 dz \neq \int_1^0 dz$, for example).

A smooth closed curve is the same as the above, except it is one-to-one for $t \in [a, b)$ but we have $z(a) = z(b)$. I.e. there is a unique z for every t except at the endpoint, where we need to close the curve on itself.

A *contour* is a slight generalization of these two ideas: it is just a finite sequence of smooth arcs that are glue together at the end point. I.e. a contour γ is just a list of smooth arcs $\{\gamma_1, .., \gamma_n\}$ with the property that $z(b_{\gamma_k}) = z(a_{\gamma_{k+1}})$. Contours make sense for integration in a way similar to integrating over a discontinuous function: integration is still well defined as long as the curve doesn't have an infinite number of sharpe corners, just like how integrating a discontinuous function is fine if there are only finitely many discontinuities.

7.2 Contour Integration

Integrating over the contour is now defined in the obvious way:

$$\int_\gamma dz f(z) \equiv \int dt \frac{\partial z_\gamma(t)}{\partial t} f(z_\gamma(t)) \tag{7.2}$$

In other words, we can just define a new function $g(t) = z'(t)f(z(t)) = u_g(t) + iv_g(t)$ and then the integral is just

$$\int dt g(t) = \int dt u_g(t) + i \int dt v_g(t) \ , \tag{7.3}$$

so that complex integration is just the sum of two real integrals. On any smooth arc, we can then use basic results from calculus of one variable. E.g. if a complex function $f(t) = F'(t)$ on all $t \in [a, b]$ then we cal also use the fundamental theorem of calculus to evaluate a single arc:

$$\int_a^b f(t)dt = F(a) - F(b) \tag{7.4}$$

We can use this to evaluate some integral easily.

This is obviously well defined on a smooth directly arc, and we can define it on a contour as just the sum over all the arcs in the contour.

$$\int_\gamma dz f(z) = \int_{\gamma_1} dz f(z) + \int_{\gamma_2} dz f(z) + \ldots + \int_{\gamma_n} dz f(z) \tag{7.5}$$

where $\gamma = \{\gamma_1, .., \gamma_n\}$ is a contour (we will use this kind of label a lot). One caveat is that we will have to be careful about using our intuition from calculus on the full contour γ rather than just the individual pieces of the curve.

Finally, note that if our function is analytic, we have a well defined z-derivative so if there is an analytic function $F(z)$ such that $f(z) = F'(z)$ everywhere, then

$$\int_\gamma dz f(z) = F(z(b)) - F(z(a)) \ . \tag{7.6}$$

As a result, if you know the indefinite integral for a function $f(x)$, chances are you can still integrate function over a contour just by plugging in the values at the endpoints. However, remember, we are not promised this will work when $f(z)$ is not analytic everywhere (we will see shortly what to do in those cases).

Example

Let's start with some very simple examples. Imagine we want to integrate the function e^z from $z = 0$ to $z = i$. We define the contour as $z = it$ for $t \in [0, 1]$ and now we are just evaluating

$$\int_\gamma dz e^z = \int_0^1 dt z' e^z = i \int_0^1 dt e^{it} = i[\frac{1}{i}(e^i - e^0)] = e^i - 1 \ . \tag{7.7}$$

This is, of course, the same result we would have function using $F(z) = e^z$ and computing

$$\int_\gamma dz e^z = F(i) - F(0) = e^i - 1 \tag{7.8}$$

However, it is useful to check it with the explicit parameterization for practice for functions that are not everywhere analytic.

Now let us do a more interesting example: integrate $(z - z_0)^n$ for some integer n around the circle $|z - z_0| = r$. So we can write the contour as $z = z_0 + re^{it}$ where $t \in [0, 2\pi]$:

$$\begin{aligned} \int_{C_r} dz(z - z_0)^n &= \int_0^{2\pi} dt(ire^{it})(re^{it})^n \\ &= r^{n+1} \int_0^{2\pi} dt e^{i(n+1)t} \end{aligned} \tag{7.9}$$

If $n \neq -1$, the we can easily see that

$$\int_{C_r} dz(z-z_0)^n = ir^{n+1}\frac{1}{i(n+1)}(e^{i2\pi(n+1)}-1) = 0 \tag{7.10}$$

However, when $n = -1$ we have

$$\int_{C_r} dz(z-z_0)^{-1} = i\int_0^{2\pi} dt = 2\pi i \tag{7.11}$$

This is a somewhat strange result: integrating in a circle around any power of z is zero, except when it's a simple pole. When it is a simple pole, the answer doesn't depend on the radius of the circle and is just $2\pi i$. Remember this funny fact, it will be related to something very important!

One thing to notice is that, even in the case of a circle, there is still a meaning to the direction of the curve. E.g. it was important I went counter-clockwise around the circle. If I had gone clockwise, $z = e^{-it}$ then I would get

$$\int_{C_r,\text{clockwise}} dz(z-z_0)^{-1} = \int_0^{2\pi} dt(-ire^{-it})(re^{-it})^{-1} = -2\pi i \tag{7.12}$$

Path Independence

Now we have arrived at the first powerful theorem: path independence. In short, the theorem will say that for sufficiently nice functions, the result of integrating along a contour sum z_1 to z_2 doesn't depend on the space of the countour. This should be reminiscent of electro-statistics where the work done by an electro-static field is independence of how you move the charge : it is just the voltage difference between the two points.

Theorem: *if $f(z)$ is a continous function on D with an anti-derivative such that $F'(z) = f(z)$ everywhere in D, then for every contour in D,*

$$\int_\gamma dzf(z) = F(z_2) - F(z_1) \tag{7.13}$$

With a small amount of work, one can show that if $f(z)$ is continuous in D, having an anti-derivative is equivalent to saying the integral is independent of path. This second statement is useful because it is sometimes easier to prove that the integrals are independent of path than it is to show that that have an anti-derivative.

7.3 Cauchy's Integral Theorem

Now we get to the fun part of complex integration: we will see that being an analytic function is sufficient to prove path independence. From there, lots of nice results follow almost immediately.

We first need the idea that two closed curves γ_0 and γ_1 are *continuously deformable* if there exists a continous function $z(s,t)$ with $s \in [0,1]$ such that $z(0,t) = \gamma_0$ and $z(1,t) = \gamma_1$ (see Figure 8). From here, we can now show that if $f(z)$ is analytic, then

$$\int_{\gamma_0} dzf(z) = \int_{\gamma_1} dzf(z) \tag{7.14}$$

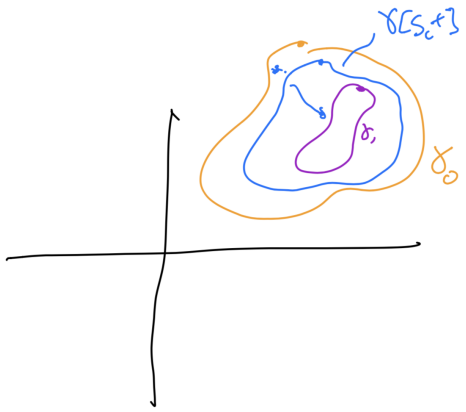

Figure 8: Deformable contours.

where γ_0 and γ_1 are continuously deformable into one another. We can define

$$I(s) = \int_a^b dt f(z(s,t)) \frac{\partial}{\partial t} z(s,t) \tag{7.15}$$

Showing equality just means that $\frac{\partial}{\partial s} I(s) = 0$, so we write

$$\begin{aligned}
\frac{\partial}{\partial s} I(s) &= \int_a^b dt \frac{\partial}{\partial s} [f(z(s,t)) \frac{\partial z}{\partial t}] && (7.16)\\
&= \int_a^b dt [f'(z(s,t)) \frac{\partial z}{\partial s} \frac{\partial z}{\partial t} + f(z(s,t)) \frac{\partial^2 z}{\partial s \partial t}] && (7.17)\\
&= \int_a^b dt [f'(z(s,t)) \frac{\partial z}{\partial t} \frac{\partial z}{\partial s} + f(z(s,t)) \frac{\partial^2 z}{\partial t \partial s}] && (7.18)\\
&= \int_a^b dt \frac{\partial}{\partial t} [f(z(s,t)) \frac{\partial z}{\partial s}] && (7.19)\\
&= [f(z(s,b)) \frac{\partial z(s,b)}{\partial s} - f(z(s,a)) \frac{\partial z(s,a)}{\partial s}] && (7.20)\\
&= 0 && (7.21)
\end{aligned}$$

In the last step we used the fact that each loop is closed so that the endpoints of the integration are always the same point in the complex plane, i.e. $z(s,a) = z(s,b)$ for all s. You can also see that this applied to smooth arcs that are deformable to each other, if we hold the endpoints fixed, $z(s,a) = z_a$ and $z(s,b) = z_b$ for all s, so that $\frac{\partial}{\partial s} z(s,a) = \frac{\partial}{\partial s} z(s,b) = 0$.

A simple consequence of this is *Cauchy's integral theorem*: If D is simply connected (i.e. all closed curves can be continuously deformed to a point), then

$$\int_\gamma dz f(z) = 0 \tag{7.22}$$

for every closed γ in D.

Next follows *Cauchy's integral formula:* for any closed positively oriented (i.e. contour clockwise) curve γ with z_0 inside the curve and an analytic function $f(z)$, then

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma dz \frac{f(z)}{z - z_0} \tag{7.23}$$

Proof: consider a circle of radius ϵ enclosing z_0. Since $f(z)$ is analytic, we can expand in a Taylor series

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \dots \tag{7.24}$$

Now we integrate each term in the sum:

$$\frac{1}{2\pi i}\int_{C_\epsilon} dz \frac{f(z)}{z - z_0} = \sum_{n=-1}^{\infty} \frac{1}{(n+1)!}\frac{d^{n+1}}{dz^{n+1}} f(z)|_{z=z_0} \int_{C_\epsilon} dz (z - z_0)^n \tag{7.25}$$

But we already know that this integral is zero except when $n - 1$ and therefore

$$\frac{1}{2\pi i}\int_{C_\epsilon} dz \frac{f(z)}{z - z_0} = \frac{1}{2\pi i} f(z_0) \int_{C_\epsilon} dz (z - z_0)^{-1} = f(z_0) \tag{7.26}$$

Now we use the fact that this integral is the same for every curve that can be continuously deformed to C_ϵ and were done.

The above proof makes the following generalization obvious:

$$\frac{d^n}{dz^n} f(z)|_{z=z_0} = \frac{n!}{2\pi i}\int_\gamma dz \frac{f(z)}{(z - z_0)^{n+1}} \tag{7.27}$$

Proof: Repeat the above proof, but now isolate the n-th term in the Taylor expansion so that the total power of $(z - z_0)$ is -1.

7.4 Cauchy's Residue Formula

Wikipedia: Residue
Wikipedia: Residue Theorem

We have seen that the problem of solving a number of integrals can be reduced to identifying the piece in a series around $z = z_0$ that behaves like $1/(z - z_0)$. But now we would like to see what it means for computing an integral $\int dz f(z)$ and $f(z)$ itself has poles.

Recall that the $f(z)$ has a pole of order n at $z = z_0$ if $f(z)$ and $f(z)(z - z_0)^m$ are not analytic at $z = z_0$ for $m < n$ but $g(z) = f(z)(z - z_0)^n$ is analytic at z_0. Locally, we see that

$$\int_{C_\epsilon} dz f(z) = \int_{C_\epsilon} dz \frac{g(z)}{(z - z_0)^n} = \frac{2\pi i}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}} g(z)|_{z=z_0} \tag{7.28}$$

It is therefore useful to define the *residue* as

$$\mathrm{Res}(f(z), z_0) = \lim_{z \to z_0} \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}[(z - z_0)^n f(z)] \tag{7.29}$$

It is then easy to see *Cauchy's Residue Theorem*: if $f(z)$ is analytic except at poles z_i then

$$\int_\gamma dz f(z) = 2\pi i \sum_i \mathrm{Res}(f, z_i) \tag{7.30}$$

Thus we have replaced the problem of integration with the problem of finding the poles of $f(z)$. Note that we are taking γ to be counter-clockwise (positively oriented). The clockwise orientation would just give a $-2\pi i \sum_i \text{Res}(f, z_i)$ on the RHS instead. (see Figure 9)

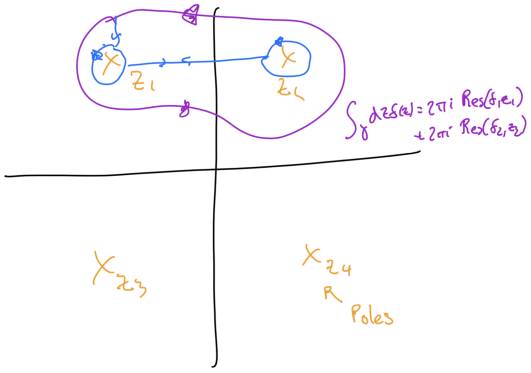

Figure 9: Illustration of the residue theorem.

Take-away: we began our discussion of holomorphic / analytic function by pointing out that they a lot like the potential for an electrostatics problem. In this analogy, we should think of poles like point sources and their residues are their electric change. Cauchy's Residue formula is like the statement of Gauss's law that the integral of the electric flux through a volume is given by the sum of the charges.

Example: First let us see how this works in a couple quick examples: Let's imagine we integrate the function $f(z) = z^3/(z^2+1)$ in a small counter-clockwise contour γ around $z = +i$. The shape of this contour doesn't matter but it is important that it encloses a pole at $z = +i$. If we write $z^2 + 1 = (z+i)(z-i)$ we can see that this is a simple pole (see Figure 10, left panel). We now have

$$\int_\gamma dz f(z) = \int_\gamma dz \frac{z^3}{(z+i)(z-i)} = 2\pi i \text{Res}(f(z), z = i) = 2\pi i \frac{z^3}{z+i}|_{z=i} = 2\pi i \frac{i^3}{2i} = -\pi i \quad (7.31)$$

Now let us perform an integral over the same contour but now with $f(z) = z^3/(z-i)^2$ (see Figure 10, right panel). We now have a second order pole but we can still apply the residue formula

$$\int_\gamma dz \frac{z^3}{(z-i)^2} = 2\pi i \text{Res}(f(z), z = i) = 2\pi i \frac{d}{dz} z^3|_{z=i} = 2\pi i 3(i)^2 = -6\pi i \ . \quad (7.32)$$

This is makes evaluating these contour integrals pretty easy. The question we will still have to address is how to make this useful to integrals that you actually want to do.

Application: We now know everything we need to resolve source of confusion we encountered with the fourier transform. When integration a Gaussian, I made the claim that

$$\int_{-\infty - ik\sigma}^{\infty - ik\sigma} dy e^{-y^2/2\sigma^2} = \int_{-\infty}^{\infty} dy e^{-y^2/2\sigma^2} = \sqrt{2\pi}\sigma \ . \quad (7.33)$$

We can prove this result by adding forming a closed contour that runs over the line segments $[-L, L]$, $[L, L - ik\sigma]$, $[L - ik\sigma, -L - i\vec{k}\sigma]$, and $[-L - i\vec{k}\sigma, -L]$ (see Figure 11). Since $e^{-z^2/2\sigma^2}$ is

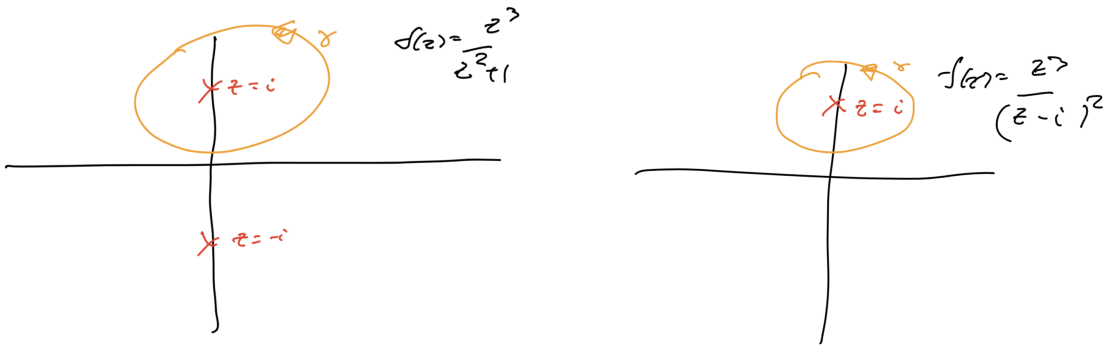

Figure 10: Examples $f(z) = z^3/(z^2+1)$ (left) and $f(z) = z^3/(z-i)^2$ (right).

an analytic function with no poles. The integral over this closed contour vanishes because there are no residues. Taking the limit $L \to \infty$, we have

$$\lim_{L\to\infty}\left(\int_{-L}^{L} dz + \int_{L}^{L-ik\sigma} dz + \int_{L-ik\sigma}^{-L-ik\sigma} dz + \int_{-L-ik\sigma}^{-L} dz\right) e^{-z^2/2\sigma^2} = 0 \tag{7.34}$$

Since

$$\lim_{L\to\infty}\int_{L}^{L-ik\sigma} dz e^{-z^2/2\sigma^2} = \lim_{L\to\infty} e^{-L^2/2\sigma^2}\int_0^1 dt e^{-(1-ik\sigma t/L)^2/2\sigma^2} \to 0 \tag{7.35}$$

we have

$$\int_{-\infty}^{\infty} dz e^{-z^2/2\sigma^2} - \int_{-\infty-ik\sigma}^{\infty-ik\sigma} dz e^{-z^2/2\sigma^2} = 0\ . \tag{7.36}$$

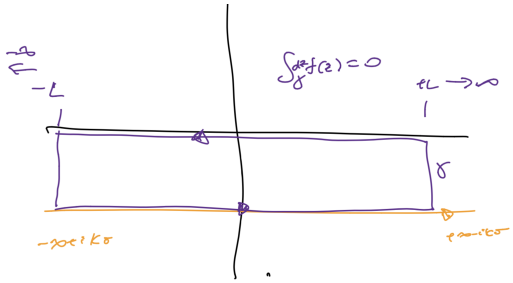

Figure 11: Contour for Gaussian integral.

7.5 The Contour at Infinity and Jordan's Lemma

Mathworld: Jordan's Lemma
Khan Academy / Youtube: Jordan's Lemma Proof

A contour of general interest is one that runs along the real axis and then closes on itself with a semi-circle at infinity. The reason we should care is that it means we can use the residue formula to write

$$\int_{-\infty}^{\infty} dx f(x) + \int_{C_\infty^+} dz f(z) = 2\pi i \sum_i \mathrm{Res}(f, z_i) \tag{7.37}$$

The first term is an integral you often want to calculate. The RHS is something that is very easy to calculate. A priori, you might think that means that the integral over C_∞^+ is equally hard as the integral we originally wanted to compute because there should be no free lunch. Amazingly, this pessimism is not justified and there are many circumstances were this integral at infinity vanishes.

The most trivial case is when $f(z) = P(z)/Q(z)$ where $P(z)$ and $Q(z)$ are polynomials of degree p and q respectively (we want them to be polynomials so that there are no branch cuts to worry about). Now we want to show that if $g \geq 2 + p$ then

$$\int_{C_\infty^+} dz f(z) = 0 \tag{7.38}$$

We can see this by taking $z = Re^{i\theta}$ and then taking $R \to \infty$:

$$\int_{C_\infty^+} dz f(z) = \lim_{R\to\infty} \int_0^\pi iRe^{i\theta} d\theta \frac{R^p e^{ip\theta}}{R^q e^{iq\theta}} \tag{7.39}$$

$$= \lim_{R\to\infty} \frac{R^{p+1}}{R^q} \int_0^\pi d\theta e^{i(p+1-q)\theta} \to 0 \tag{7.40}$$

If we didn't have to worry about branch cuts, this would show that if the $q > p + 1$ the integral will vanish as $R \to \infty$.

Jordan's Lemma: Given a function $f(z) = e^{iaz} g(z)$ where $a > 0$ and $g(z)$ is holomorphic up to a finite number of poles in the upper half plane.

$$\int_{C_R^+} dz f(z)| \leq \frac{\pi}{a} \max_{\theta=[0,\pi]}(g(Re^{i\theta})) \tag{7.41}$$

This means if $g(Re^{i\theta}) \to 0$ as $R \to \infty$ (for all $\theta \in [0, \pi]$ then the integral at infinity vanishes.

The intuition is that this should be better than just polynomials because as $z \to iy \to i\infty$, this function is $f(z) \sim e^{-ay} g(iy)$ which vanishes exponentially (see Figure 12). Of course, part of this integral is near $z \sim \pm\infty$ and there is no exponential suppression so we need to be more careful.

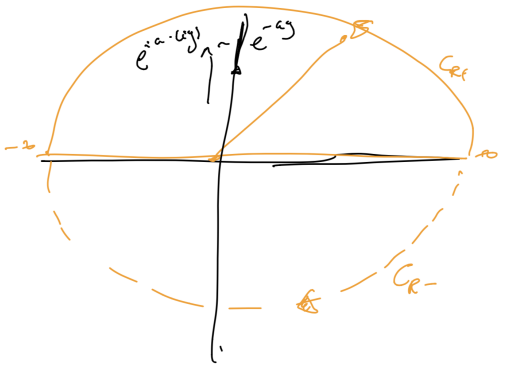

Figure 12: Contour for Gaussian integral.

Proof: We define the integral as before

$$\int_{C_r^+} dz f(z) = \int d\theta (iRe^{i\theta}) e^{iaR(\cos\theta + i\sin(\theta))} g(Re^{i\theta}) \tag{7.42}$$

First we are going to use a simple inequaly

$$|\int dx f(x)| \leq \int dx |f(x)| \tag{7.43}$$

to write

$$|\int d\theta (iRe^{i\theta}) e^{iaR(\cos\theta + i\sin(\theta))} g(Re^{i\theta})| \leq R\int d\theta e^{-aR\sin\theta} |g(Re^{i\theta})| \tag{7.44}$$

$$\leq R\text{max}_{\theta=[0,\pi]}(g(Re^{i\theta}))\int_0^\pi d\theta e^{-aR\sin\theta} \tag{7.45}$$

Because of $\sin\theta$ is symmetric about $\pi/2$ we can just integral from 0 to $\pi/2$ twice. Now as a final step, one uses $\sin\theta \geq 2\theta/\pi$ on $\theta \in [0, \pi/2]$ so make and therefore $e^{-aR\sin\theta} < e^{-a\theta/\pi}$ onto bound the integral:

$$|\int d\theta (iRe^{i\theta}) e^{iaR(\cos\theta + i\sin(\theta))} g(Re^{i\theta})| \leq 2R\text{max}_{\theta=[0,\pi]}(g(Re^{i\theta}))\int_0^{\pi/2} d\theta e^{-2aR\theta/\pi} \tag{7.46}$$

$$\leq \frac{\pi \text{max}_{\theta=[0,\pi]}(g(Re^{i\theta}))}{a}(1 - e^{-aR}) \tag{7.47}$$

Return to Causality

So let's return to our original motivation: where does the requirement of causality, $G(t) = 0$ when $t < 0$, appear in fourier space. Let suppose we have $\tilde{G}(\omega)$ and we write

$$G(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t}\tilde{G}(\omega) \tag{7.48}$$

Using Jordan's lemma, if $\tilde{G}(\omega)$ is analytic in the upper half plane and $G(\omega) \to 0$ as on C_∞^+ (as we go to infinity in the upper half plane), then $G(t) = 0$ for $t < 0$.

We can see this directly in our example from the Harmonic osclilator

$$G(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{(\omega_0^2 - \omega^2 - i\gamma\omega)} e^{-i\omega t} \tag{7.49}$$

We notice that this function has two poles (see Figure 13)

$$\omega_\pm = -i\frac{\gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} = -i\frac{\gamma}{2} \pm \bar{\omega} \tag{7.50}$$

which are both in the lower half plane. As a result, Jordan's lemma means that $G(t < 0) = 0$. For $t > 0$, we close in the lower half-plane and pick up the two poles

$$G(t) = i\left[\frac{1}{2\bar{\omega}} e^{-\frac{\gamma}{2}t} e^{-i\bar{\omega}t} - \frac{1}{2\bar{\omega}} e^{-\frac{\gamma}{2}t} e^{i\bar{\omega}t}\right] \tag{7.51}$$

$$\tag{7.52}$$

Putting this all together we have

$$G(t) = \frac{1}{\bar{\omega}} e^{-\frac{\gamma}{2}t} \sin(\bar{\omega}t)\theta(t) \tag{7.53}$$

where $\theta(t) = 1$ when $t > 0$ and 0 when $t < 0$ is a step-function.

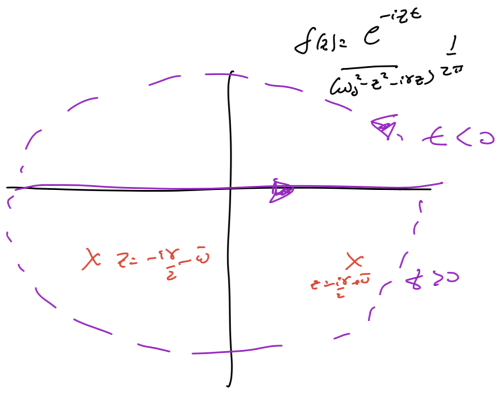

Figure 13: Contour for Green's function inverse Fourier transform.

Examples

Let's take a case we know well:

$$f(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \frac{2b}{\omega^2 + b^2} \tag{7.54}$$

when $t < 0$, we can close the contour in the upper half plane and use Jordan's lemma to ingore the piece at infinity. We now use the residue theorem with the pole at $\omega = +ib$ to get

$$f(t < 0) = 2\pi i \times \frac{1}{2\pi} \frac{2b}{2ib} e^{bt} = e^{bt} \tag{7.55}$$

When $t > 0$, we have to close in the lower half plane, and use the pole at $\omega = -ib$ to find

$$f(t > 0) = -2\pi i \times \frac{1}{2\pi} \frac{2b}{-2ib} e^{-bt} = e^{-bt} \tag{7.56}$$

Notice we had to flip the overall sign because we went from being clockwise to counter-clockwise. So we found, as we knew already that

$$f(t) = e^{-b|t|} \ . \tag{7.57}$$

Another simple example (see Figure 14):

$$\int_{-\infty}^{\infty} dx \frac{x^2 + x + 1}{(x^2+1)^2} = 2\pi i \frac{d}{dz} \frac{z^2 + z + 1}{(z+i)^2}|_{z=i} \tag{7.58}$$

$$= \pi \tag{7.59}$$

You can plug this into mathematica and find the same result.

7.6 Integrals in the Presence of Branch Cuts

The simplification of integrals is not quite the same when there are branch points along the real axis. It is easiest to see this with an example:

$$\int_0^{\infty} dx \frac{\sqrt{x}}{(x+4)^2} \tag{7.60}$$

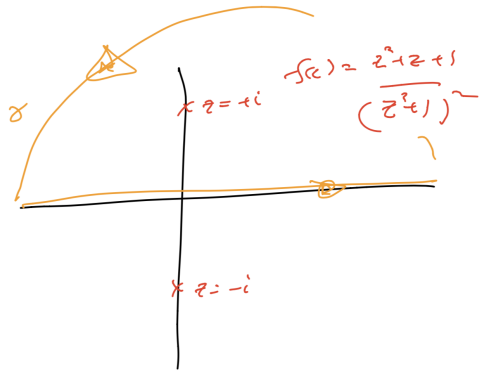

Figure 14: Contour for integrating $\int_{-\infty}^{\infty} dx \frac{x^2+x+1}{(x^2+1)^2}$.

It is helpful to put the branch on the positive axis and the draw a big loop around the branch cut and close it in a circle at infinity (see Figure 15). Then the integral breaks into a piece above the cut, a piece below the cut and a circle at infinity that vanishes. The fact that the circle vanishes is a simple generalization of our $P(z)/Q(z)$ arguement for polynomials: when $z = R^{i\theta}$ with $R \to \infty$, our integrand scale like $R^{1/2}/R^2 \propto R^{-3/2}$. The circumference of the circle scale like R so the integral over C_R will vanish like $R^{-1/2} \to 0$ as $R \to \infty$.

We can now evaluate this integral if we are careful to place the branch cut on the positive x-axis. This means we define $\sqrt{z}$ by taking $z = \rho e^{i\theta}$ where $\theta \in [0, 2\pi)$. The contour integral is therefore

$$\int_0^{\infty} dx \frac{\sqrt{x}}{(x+4)^2} + \int_{\infty}^{0} dx \frac{\sqrt{xe^{i2\pi}}}{(x+4)^2} = 2\pi i \text{Res}(\frac{\sqrt{z}}{(z+4)^2}, z = -4) \tag{7.61}$$

$$2\int_0^{\infty} dx \frac{\sqrt{x}}{(x+4)^2} = 2\pi i \frac{d}{dz}\sqrt{z}|_{z=-4} \tag{7.62}$$

$$= 2\pi i \times \frac{1}{2}\frac{1}{\sqrt{4e^{i\pi}}} \tag{7.63}$$

$$= \frac{\pi}{2} \tag{7.64}$$

or

$$\int_0^{\infty} dx \frac{\sqrt{x}}{(x+4)^2} = \frac{\pi}{4} \ . \tag{7.65}$$

This is an interesting case because it isn't just some simple application of the residue theorem. Notice that our evaluation of the residue at $z = -4$ was only well defined by because we have placed the branch cut on the positive x-axis. The residue otherwise would have been ambiguous.

There are two general lessons with these kinds of examples that you need to pay attention to: (1) The range of integration over x will run between to branch points or from a branch point to infinity. (2) The integrand must scale like $R^{-\alpha}$ with $\alpha > 1$ as $R \to \infty$. Because there is a branch cut, this doesn't have to be an integer, but it must be strictly large than 1. To see this two principles in action, we could consider

$$\int_{-1}^{-1} dx \frac{\sqrt{1-x^2}}{(x+2)^3} \tag{7.66}$$

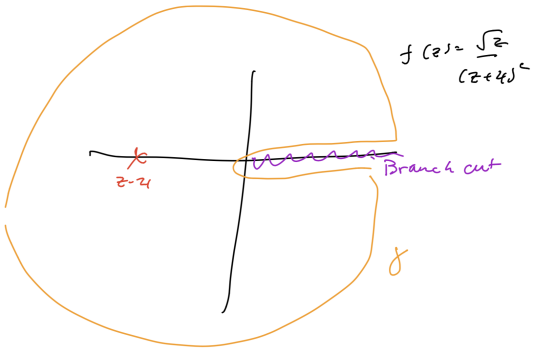

Figure 15: Contour for integrating $\int_{-\infty}^{0} dx \frac{\sqrt{x}}{(x+4)^2}$.

Repeating the same logic, we can put the branch cuts between the two branch cuts between the branch points the way we did above. Then we can see that the integrand scales as R^{-2} as $R \to \infty$ so the circle at infinity will vanish. So that we have

$$2\int_{-1}^{1} dx \frac{\sqrt{1-x^2}}{(x+2)^3} = 2\pi i \text{Res}(\frac{\sqrt{z}}{(z+2)^3}, z=-2) = 2\pi i \frac{1}{i6\sqrt{3}} \tag{7.67}$$

However, if you were to consider instead

$$\int_{-1}^{-1} dx \frac{\sqrt{1-x^2}}{(x+2)^2} \tag{7.68}$$

when the integral around the circle doesn't vanish and you find

$$2\int_{-1}^{1} dx \frac{\sqrt{1-x^2}}{(x+2)^3} + \int_{C_R} dz \frac{\sqrt{1-z^2}}{(z+2)^3} = 2\pi i \text{Res}(\frac{\sqrt{z}}{(z+2)^2}, z=-2) = 2\pi i \frac{2}{i\sqrt{3}} \tag{7.69}$$

At infinity, we we expand in $z = Re^{i\theta}$ with $R \to \infty$ then

$$\int_{C_R} dz \frac{\sqrt{1-z^2}}{(z+2)^2} \approx \int_{C_R} dz \frac{1}{z} = 2\pi i \tag{7.70}$$

Moving this term to the right hand side, we get

$$\int_{-1}^{1} dx \frac{\sqrt{1-x^2}}{(x+2)^3} = \frac{2}{\sqrt{3}}\pi - \pi \tag{7.71}$$

You can check these results with Mathematica and see this funny behavior of the integrals for yourself if you don't trust our complex analysis.

8 Ordinary Differential Equations

Dubin Chapter 1.6

So far we have been focused on the harmonic oscillator and the wave equation. These examples as particularly useful as they are (1) extremely common and describe lots of physical systems and (2) are dramatically simplified by using the fourier transform. Now we want to take the lessons we have learned from these examples and generalize them to more complicated situations. In this topic, we will focus on a generalization of the harmonic oscillator, namely linear ordinary differential equations (linear ODEs), but will introduce the ideas that will also be important for generalizations of the wave equation to linear partial differential equations (linear PDEs).

8.1 Basic Structure of an ODE

We are interested in equations that take the form:

$$\hat{L}\left(\frac{d}{dt}, t\right) x(t) = h(t) \tag{8.1}$$

where $\hat{L}(\frac{d}{dt}, t)$ is some kind of machine (we'll call it an "operator") that acts on functions. Specifically, the derivatives inside $\hat{L}$ will act on any function to the right of $\hat{L}$ but not to the left. Will assume it can be written as a polynomial in derivatives, $\frac{d}{dt}$ or some other variable of our choosing. We can organize such an equation so that all of the derivatives act on $x(t)$, which is what we are trying to solve for. The function $h(t)$ is a function we are given and is some kind of generalization of the external force. Notice that if $h(t) = 0$ then $x(t) = 0$ is a valid solution, so whatever the physical origin of $h(t)$ is, it is trying to make $x(t) \neq 0$.

Differential "Operator"

We will think of $\hat{L}(\frac{d}{dt}, t)$ as a polynomial of order n in $\frac{d}{dt}$ so that we can write

$$\hat{L}(\frac{d}{dt}, t) = w_n(t)\frac{d^n}{dt^n} + w_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}} + \ldots w_1(t)\frac{d}{dt} + w_0 \tag{8.2}$$

In general, these functions $w_i(t)$ are arbitrary functions of t (not necessarily polynomials). However, in practice, we can only solve these equations for particularly simple functions of t. Up until now, we have been interested in the cases where $n = 1, 2$ and these functions are actually just constants (i.e. they do not depend on time).

We also think of this as an "operator" that acts to the right. E.g. if $\hat{L} = \frac{d}{dt}$ then

$$\hat{L}x(t) = \dot{x}(t) \qquad x(t)\hat{L} = x(t)\frac{d}{dt} \qquad (x(t)\hat{L})y(t) = x(t)(\hat{L}y(t)) = x(t)\dot{y}(t) \tag{8.3}$$

Linearity

A operator, $\hat{L}$, is defined to be linear if it has the following properties:

$$\hat{L}(x_1(t) + x_2(t)) = (\hat{L}x_1(t)) + (\hat{L}x_2(t)) \tag{8.4}$$

$$\hat{L}(cx(t)) = c(\hat{L}x(t)) \tag{8.5}$$

where c is a constant.

Linearity is the origin of the superposition principle that appears in electrostatics and quantum mechanics. The idea is that if we have two solutions to the equation, $x_1(t)$ and $x_2(t)$ such that $\hat{L}x_1 = 0$ and $\hat{L}x_2 = 0$, then we can find a third solution $x_3 = c_1x_1(t) + c_2x_2(t)$. This is a simple application of linearity:

$$\hat{L}(c_1x_1(t) + c_2x_2(t)) = c_1\hat{L}x_1 + c_2\hat{L}x_2 = 0 \ . \tag{8.6}$$

Given an equation that is a polynomial of order n in ∂_t, there are typically n-solutions, $x_i(t)$ for $i = 1, .., n$ such that

$$\hat{L}(\tfrac{d}{dt}, t)x_i(t) = 0 \tag{8.7}$$

Because the equation is linear we can write the most general solution as

$$x(t) = \sum_{i=1}^{n} c_i x_i(t) \ . \tag{8.8}$$

Now, it isn't always true that there are n-independent solutions so this isn't a rigorous result but just a general expectation (e.g. you know from electrostatics that there is a unique solution to the potential for a change distribution, up to an overall constant).

We call the solutions to the equation $\hat{L}x(t) = 0$ the solutions to the homogenous equation. If we want to solve the equation $\mathcal{L}x(t) = h(t)$ then the most general solution will take the form

$$x(t) = x_{\text{inhom.}}(t) + \sum_{i=1}^{n} c_i x_i(t) \ . \tag{8.9}$$

where $x_{\text{inhom.}}(t)$ is any solution to the equation $\hat{L}x_{\text{inhom.}}(t) = h(t)$. You might think this is too restrictive because there can be multiple solutions to this equation, $x_{j,\text{inhom.}}(t)$ for $j = 1, .., m$ such that $\hat{L}x_{j,\text{inhom.}}(t) = h(t)$. However, you can see that were aren't missing any solutions as follows: suppose that you take $x_{\text{inhom.}}(t) = x_{j=1,\text{inhom.}}(t)$ and I take $x_{\text{inhom.}}(t) = x_{j=1,\text{inhom.}}(t)$ and look at the difference of our two solutions $\Delta x_{\text{inhom}}(t) = x_{j=1,\text{inhom.}}(t) - x_{j=2,\text{inhom.}}(t)$. Using linearity, we see that

$$\hat{L}\Delta x_{\text{inhom}} = \hat{L}x_{j=1,\text{inhom.}}(t) - \hat{L}x_{j=2,\text{inhom.}}(t) = h(t) - h(t) = 0 \tag{8.10}$$

Therefore $\Delta x_{\text{inhom}}(t) \neq 0$ is just some solution to the homogenous equation! Therefore, the relationship between our two solutions is just that we have different coefficients c_i. I.e.

$$\Delta x_{\text{inhom}}(t) = \sum_{i=1}^{n} \Delta c_i x_i(t) \tag{8.11}$$

so that

$$\begin{aligned} x(t) &= x_{j=1,\text{inhom.}}(t) + \sum_{i=1}^{n} c_i x_i(t) && (8.12)\\ &= x_{j=2,\text{inhom.}}(t) + \Delta x_{\text{inhom}} + \sum_{i=1}^{n} c_i x_i(t) && (8.13)\\ &= x_{j=2,\text{inhom.}}(t) + \sum_{i=1}^{n} (c_i + \Delta c_i) x_i(t) && (8.14) \end{aligned}$$

Therefore we only ever need to find a single solution to the inhomogenous equation.

We can use the same trick of linearity to find solutions to $h(t) = \sum_q d_q f_q(t)$. In particular, if we can solve

$$\hat{L}x_q(t) = f_q(t) \tag{8.15}$$

for some particular $x_q(t)$, then we can solve the full equation using

$$x(t) = \sum_q d_q x_q(t) \tag{8.16}$$

We can see this just by plugging into our equation:

$$\begin{aligned} \hat{L}x(t) &= \hat{L}\sum_q d_q x_q(t) && (8.17)\\ &= \sum_q d_q \hat{L}x_q(t) && (8.18)\\ &= \sum_q (d_q f_q(t)) && (8.19)\\ &= h(t) && (8.20) \end{aligned}$$

This will be a useful way to think about Green's functions.

8.2 Boundary Conditions and Initial Value Problems

In order to determine the constants c_n that sit in our list of homogenous solutions, we need to decide what properties we want out solution to have. There are three basic ways we can do this:

- Fix the value of a function at a point: e.g. $x(t = 0) = 0$
- Fix the value of a derivative at a point: e.g. $\dot{x}(t = 0) = 1$
- Fix the behavior at $\pm\infty$: E.g. if our solution has the form $\phi(x) = e^{\pm\rho x}$ we might require that $\phi(x \to \infty)$ is finite.

For a typical second order differential equation, we need to impose two conditions to fix the two constants that typically arise.

There are typically two different ways these conditions can be deployed

- *Initial value problems:* In this case we are imposing all of our constraints at a specific point (we will interpret as a point in time) and as what happens in the future. E.g. we might give the initial position $x(0)$ and velocity $\dot{x}(0)$ and ask there the particle moves in the future. These are also easy to evaluate numerically because you are can essentially just integrate the equation one time step at a time to figure out how the particle evolves from $x(0)$ and $\dot{x}(0)$ to any other configuration.
- *Boundary value problems:* rather than imposing all the conditions at one point, we impose conditions at multiple points. E.g. I might want to know the shape of a piece of string, $h(x)$, it I hold it at spefic points at the ends, $h(0) = h_1$ and $h(L) = h_2$. If I impose a condition

on the function at that point ($x(0) = 5$) is is a Dirichlet boundary condition and if the condition is on the derivate ($\dot{x} = 5$) then it is called a Neumann boundary condition. You can also have a linear combination of the two $c_1 x(0) + c_2 \dot{x}(0) = c_3$ that are called Robin boundary conditions. (Technically there are 2 end points and these names usually refer to applying the same type at both ends and *mixed* boundary conditions refer to applying different choices of these three at each end.)

- The case where we impose behavior at $x = \pm\infty$, it is best to think of an analytic solution of this type as just a boundary value problem. However, numerically, it can be very difficult to impose the kinds asymptotic conditions.

- If we are only solving the equation on some domain $t \in [0, L]$ (for example) then we could similarly have the boundary condition that we want the solution that is well behaved at the boundaries. E.g. if we had solutions of the form $(a \cos x + b \sin x)/x$ then we might want to take $a = 0$ so that the solution doesn't diverge as $x \to 0$.

8.3 Power-Law Solutions

The simplest situations to solve after exponentials are those where the solutions are power-laws, like t^γ for some fixed γ. This is best seen by example:

$$at^2\ddot{x}(t) + (b + a)t\dot{x}(t) + cx(t) = 0 \tag{8.21}$$

where a, b, c are some constants. If we plug in the ansatz, $x(t) = Ct^\gamma$ we get

$$Ct^\gamma(a\gamma(\gamma - 1) + (b + a)\gamma + c) = 0 \tag{8.22}$$

Notice that this implies

$$a\gamma^2 + b\gamma + c = 0 \tag{8.23}$$

so that γ is a solution of the quadratic equation.

Why did this work? Notice that if we think of $\frac{d}{dt}$ as counting like t^{-1} all the terms have the same number of powers of t in them. As a result, when we plug in our ansatz, we get a polynomial in γ but we don't change the power of t. As a result, we can factor our t^γ and just set the polynomial to zero.

The number of examples where this works are limited, but the intuition behind the power law is very important. In fact, it underlies the series solution which applies much more broadly, as we will see.

Physically, there actually a very deep reason this gives a power law. By the same observation above, notice that if I rescale time $t = \lambda t'$ then $\frac{d}{dt} = \lambda^{-1}\frac{d}{dt'}$. In the above example this means that

$$\hat{L}(\tfrac{d}{dt}, t) = \hat{L}(\tfrac{d}{dt'}, t') \tag{8.24}$$

We will find power law solutions when a weak condition is met, namely

$$\hat{L}(\tfrac{d}{dt}, t) = \lambda^\Delta \hat{L}(\tfrac{d}{dt'}, t') \tag{8.25}$$

In particular, we notice that the equation does not change as we rescale time. As a result, the solution has to look the same on all time scales. I.e. it should look the same over a few nanoseconds or a few billion years. This physical property is known as *scale invariance* and is essential to the understand a vast set to physical phenomena, from phase transitions and chaos to the origin of structure in the universe and the quantum nature of gravity. It is even more essential as an approximation to the dynamics in a wider range of systems.

Given scale invariance, we find power law solutions because they are essentially the only functions of 1 variable that obey this property. In contrast, for the harmonic oscillator we have $x(t) = e^{i\omega t}$, it looks like a constant if $t \ll 1/\omega$ and oscillates when $t > 1/\omega$. One easy way to think about this is that ω is a constant with dimensionful units (i.e. it has units $1/[\text{time}]$). This means the problem has a natural time scale associated with it. Power-law solutions are what we find when things are scale invariant, as in there is no special dimensionful scale in the problem.

8.4 Series Solutions

Most ODEs do not have solutions in terms of simple functions. The typical experience is if plug even a somewhat non-trivial ODE into mathematica, if it can solve it, it will spit out a solution in terms of functions with some special name (e.g. Bessel, Hermite, Hypergeometric, etc.). In some cases, these functions are literally defined to by the fact that they are solutions to some differential equation, so what exactly does this mean?

The most concrete way to understand this is to think about the function as a series expansion. E.g. if you aren't sure what e^{ix} means, you can simply define it as

$$e^{ix} \equiv 1 + (ix) + \frac{1}{2}(ix)^2 + \frac{1}{3!}(ix)^3 + \dots \ . \tag{8.26}$$

The series expansion of the exponential actually converges for all x so simply defining the function by the series is actually a rigorous definition.

As a simple warm-up, we should be able to see how the series describing the exponential solves ODEs like the simple harmonic oscillator:

$$\ddot{x}(t) + \omega_0^2 x(t) = 0 \ . \tag{8.27}$$

If we guess there is a series solution of the form $x(t) = t^\gamma \times \sum_{n=0}^{\infty} c_n t^n$ and plug it into the equation, we get

$$\sum_n ((n+\gamma)(n+\gamma-1)t^{n+\gamma-2}c_n + \omega_0^2 t^{n+\gamma}c_n) = 0 \tag{8.28}$$

We need to satisfy this equation order by order in t because c_n and γ must be t-independent. Starting from the smallest power of t we have

$$\sum_n t^{\gamma+n-2}((n+\gamma)(n+\gamma-1)c_n + \omega_0^2 c_{n-2}) = 0 \tag{8.29}$$

and therefore

$$(n+\gamma)(n+\gamma-1)c_n + \omega_0^2 c_{n-2} = 0 \tag{8.30}$$

By definition $c_{-2} = c_{-1} = 0$. Therefore, to solve this equation for $n = 0$ we have

$$(\gamma)(\gamma-1) = 0 \tag{8.31}$$

Our options are therefore $\gamma = 0$ or $\gamma = 1$. Since our equation jumps 2, we can solve this equation for the odd and even power separately.

Let's start with the even so that $c_0 \neq 0$ and $\gamma = 0$. Using the recursive formula, we have

$$c_2 = -\omega_0^2 \frac{c_0}{2 \times 1} \qquad c_4 = -\omega_0^2 \frac{c_2}{4 \times 3} = \frac{c_0 \omega_0^4}{4!} .. \qquad c_{2m} = (-1)^m \frac{c_0 \omega_0^{2m}}{(2m)!} \tag{8.32}$$

Similarly, if we start from $n = 1$, we have $c_{2m+1} = (-1)^m c_1 \omega_0^{2m+1}/((2m+1)!)$. The even terms are just the expansion of $c_0 \cos \omega_0 t$ and the odd terms are $c_1 \sin \omega_0 t$. We knew the solutions ahead of time, but what might not have been obvious is that the solution would break up into sines and cosines rather than $e^{i\omega_0 t}$.

We can think of most special functions in the same way. To see this, let's just consider the example of the Bessel function $J_\alpha(t)$ and $Y_\alpha(t)$ which are defined to be the two solutions to the equation

$$[t^2 \frac{d^2}{dt^2} + t \frac{d}{dt} + (t^2 - \alpha^2)]x(t) = 0 \tag{8.33}$$

Now let us assume that $x(t) = t^\gamma \times \sum_{n=0}^{\infty} c_n t^n$ (I pulled out an overall t^γ for some unknown γ in case the solution as $t \to 0$ is a non-integer power law (this is based on the educated guess that as $t \to 0$, $(t^2 - \alpha^2) \to -\alpha^2$ and this equation looks like one we can solve with a fixed power law). So, let's just plug this series into the equation

$$t^\gamma \sum_n (t^n(\gamma + n)(\gamma + n - 1)c_n + c_n(\gamma + n) - c_n \alpha^2 + c_{n-2}) \quad = \quad 0 \tag{8.34}$$

$$t^\gamma \sum_n t^n \left((\gamma^2 - \alpha^2 + 2n\gamma + n^2)c_n + c_{n-2}\right) = 0 \tag{8.35}$$

where $c_{n-2<0} = 0$ by construction. Since these coefficients are time independent, each term in the series must vanish seperately if the equation is going to be solved for all t. For $n = 0$ and $n = 1$, $c_{n-2} = 0$ and we can easily solve:

$$n = 0 \quad \to \quad \gamma^2 = \alpha^2 \to \gamma = \pm\alpha \tag{8.36}$$

$$n = 1 \quad \to \quad c_1 = 0 \tag{8.37}$$

For $n \geq 2$ now we can use the equation to determine c_n in terms of c_{n-2}. Since $c_1 = 0$, this means $c_{n=\text{odd}} = 0$. The even one are given by a recursion relation

$$c_n = -\frac{1}{\pm 2n\alpha + n^2} c_{n-2} \to c_n = c_0 \prod_{m=2,4,..}^{n} \frac{-1}{\pm 2m\alpha + m^2} \tag{8.38}$$

For example, for $n = 2$ and $\alpha = 1/2$ and $\gamma = \pm 1/2$ we have

$$c_2 = -c_0 \frac{1}{4 \pm 2} \tag{8.39}$$

If you look at the series expansion of $J_\alpha(t)$ and $Y_\alpha(t)$ for $\alpha = 1/2$ we have

$$J_{1/2}(t) \quad \approx \quad \sqrt{\frac{2}{\pi}}[t^{1/2} - \frac{1}{6}t^{5/2} + \ldots] \tag{8.40}$$

$$Y_{1/2}(t) \quad \approx \quad -\sqrt{\frac{2}{\pi}}[t^{-1/2} - \frac{1}{2}t^{3/2} + \ldots] \tag{8.41}$$

so we see that $\gamma = +1/2$ corresponds to $J_{1/2}(t)$ with $c_0 = \sqrt{\frac{2}{\pi}}$ and $\gamma = -1/2$ is $Y_{1/2}$ with $c_0 = -\sqrt{\frac{2}{\pi}}$. This property that $\gamma = +\alpha$ $(-\alpha)$ corresponds to J_α (Y_α) holds in general. The relationship between c_0 is more complicated but this is clearly no important since we always write the solution as

$$x(t) = C_1 J_\alpha(t) + C_2 Y_\alpha(t) \ . \tag{8.42}$$

This way of thinking about solutions to differential equations is the simplest general way I know to understand what these solutions are (other than numerically solving them). Of course, you can always use mathematica to provide the solution in terms of functions that mathematica knows how to evaluate, but in the end, the only reliable way to understand why these functions are solutions at all is to go back to the series expansions. In specific cases, these functions will have other definitions that also allow you to see why they solve the equation but this works only on a case by case basis.

8.5 Green's Functions

We want to understand how to think about Green's functions in this general context. The general setup is that we are looking for a function of two variables $G(t, t')$ so that

$$\hat{L}_t G(t, t') = \delta(t - t') \tag{8.43}$$

First, we can just check that if this is true, we have a solution to the equation

$$x(t) = \int_{-\infty}^{\infty} dt' G(t, t') h(t') \to \hat{L}x(t) = \int_{-\infty}^{\infty} dt' (\hat{L}G(t, t')) h(t') = \int_{-\infty}^{\infty} dt' \delta(t - t') h(t') = h(t) \tag{8.44}$$

We also remember that once we have one solution to this equation, every other solution is just related by adding the homogenous solutions.

It may also be useful to think of these expression in terms of linear algebra: if we think of $\hat{L}$ as being like a matrix, then this equation is like an equation of the matrix inverse $G = \hat{L}^{-1}$ so that $LG = I$. Given the inverse, if you wanted to solve the equation $L\vec{x} = \vec{h}$, you apply multiply by L^{-1} so that

$$\vec{x} = L^{-1}\vec{h} \equiv G\vec{h} \ . \tag{8.45}$$

This is clear in fourier space in situations where $\hat{L} \to L(\omega)$ is just a polynomial in the frequency, ω. In those circumstances, $G(t, t') = G(t - t')$ and therefore $\tilde{G}(t, t') \to \tilde{G}(\omega, \omega') = \tilde{G}(\omega)\delta(\omega + \omega')$ is diagonal in ω and solves the equation

$$L(\omega)\tilde{G}(\omega) = 1 \tag{8.46}$$

as you would expect for the diagaonal entries of the matrix equation $LG = I$.

It is worth noting that this function is really a function of two variables t and t', unlike the cases we have solved with the fourier transform. Recall what we found in that case,

$$\tilde{x}(\omega) = \tilde{G}(\omega)\tilde{h}(\omega) \to x(t) = \int dt' G(t - t') h(t') \tag{8.47}$$

Here we see that $G(t,t') = G(t-t')$ is really us a function of one variable $t-t'$. This is a consequence of the fact that the $\hat{L}_t G(t,t') = \delta(t-t')$ doesn't depend on t itself but only derivatives or the difference $t-t'$, so that if we shifted $t \to t+c$ and $t' \to t'+c$ the equation is the same. As a result, $G(t,t')$ could only depend on this combination.

For a general linear ODE, the coefficients in the equation can depend on t but cannot depend on t'. Therefore, we really have to treat t and t' independently. It is somewhat important therefore to remember that t is the time that appears in the equation and t' this the time of the source. For completeness, let's see how to solve for $G(t,t')$ without using the fourier transform. Let's take

$$[\frac{d^2}{dt^2} + \omega_0^2]G(t,t') = \delta(t-t') \tag{8.48}$$

we happen to know how to solve this using the fourier transform but let's pretend we don't and see if we can get the same answer. This first thing to notice that is that when $t \neq t'$ this equation is just

$$[\frac{d^2}{dt^2} + \omega_0^2]G(t,t')|_{t\neq t'} = 0 \tag{8.49}$$

Therefore, we know that both $G(t<t',t')$ and $G(t>t',t')$ are just solutions to the homogenous equations $e^{\pm i\omega_0 t}$, so we have

$$\begin{aligned} G(t>t',t') &= A(t')e^{i\omega_0 t} + \text{c.c.} & (8.50)\\ G(t<t',t') &= B(t')e^{i\omega_0 t} + \text{c.c.}\ . & (8.51) \end{aligned}$$

Notice that the "constant" can be a function of t', it just can't depend on t. Now the problem of solving the equation is just how to determine $A(t')$ and $B(t')$ so that we get the $\delta(t-t')$ on the RHS of the equation. First, we are interested in the *retarded* Green's function, so that $G(t<t',t') = 0$ (i.e. cause does not come before the effect). This means we are taking $B(t') = 0$.

To figure out what $A(t')$ should be, it is useful to integrate the equation over a tiny integral around t', $t = t'-\epsilon \to t'+\epsilon$:

$$\begin{aligned} \int_{t'-\epsilon}^{t'+\epsilon} dt[\frac{d^2}{dt^2} + \omega_0^2]G(t,t') &= \int_{t'-\epsilon}^{t'+\epsilon} dt\delta(t-t') & (8.52)\\ \frac{d}{dt}G(t,t')|_{t'-\epsilon}^{t'+\epsilon} + \int_{t'-\epsilon}^{t'+\epsilon} dt\omega_0^2 G(t,t') &= 1 & (8.53) \end{aligned}$$

The RHS integrated to 1 because it is a δ-function. Now the LHS is might not be a priori obvious what to do. It seems physically reasonable that we should want $G(t,t')$ to be continuous at $t=t'$ (presumably $G(t',t')$ should be well-defined). With this assumption, $\int_{t'-\epsilon}^{t'+\epsilon} dt\omega_0^2 G(t,t') = 0$ (actually, this is true even without this assumption, but it is very obvious is the function is continuous). Since $G(t<t',t') = 0$ so is $\partial_t G(t<t',t') = 0$ so combining the continuity and our equation we have two conditions:

$$\begin{aligned} A(t')e^{i\omega_0 t'} + \text{c.c.} &= 0 & (8.54)\\ \frac{d}{dt}A(t')e^{i\omega_0 t}|_{t=t'} + \text{c.c.} = i\omega_0 A(t')e^{i\omega_0 t'} + \text{c.c.} &= 1 & (8.55) \end{aligned}$$

These are sufficient to solve for the real and imaginary parts of A. If we add the two equations:

$$A(t') = -A^*(t')e^{-i2\omega_0 t'} \tag{8.56}$$

$$A(t') = A^*(t')e^{-i2\omega_0 t'} + \frac{e^{-i\omega_0 t'}}{i\omega_0} \tag{8.57}$$

we have

$$A(t') = \frac{e^{-i\omega_0 t'}}{i2\omega_0} \tag{8.58}$$

so

$$G(t < t', t') = \frac{e^{-i\omega_0 t'}}{i2\omega_0} e^{i\omega t} + c.c. \tag{8.59}$$

$$= \frac{1}{\omega_0} \sin(\omega_0(t - t')) \ . \tag{8.60}$$

It is easy to see this satisfies our requirements:

- $G(t \to t', t') \to 0$ so it is continuous at $t = t'$.
- $G(t, t') \sim (t - t')$ as $t \to t'$ so that $\frac{d}{dt} G(t, t')|_{t=t'} = 1$. This was need to match the δ-function after integrating the equation.
- We also derived that $G(t, t') = G(t - t')$ without assuming it ahead of time, which is a nice check that this method gives the same answer as using the fourier transform.

8.6 Using the Fourier Transform

As a general rule, it is very useful when you can find a list of functions x_λ such that

$$\hat{L} x_\lambda = f(\lambda) x_\lambda \tag{8.61}$$

where $f(\lambda)$ is some function of a parameter (or list of parameters). This is exactly what we did with the fourier transform where $\lambda = \omega$ and $f(\omega)$ was some polynomial. What made this possible is that our equation had constant coefficients, i.e.

$$w_n(t) \frac{d^n}{dt^n} = w_n \frac{d^n}{dt^n} \ . \tag{8.62}$$

The general reason is that $\partial_t \to -i\omega$ under fourier transform but $t \to -i\partial_\omega$. This first identify is just integration by parts and the second one is

$$\int dt e^{i\omega t} t f(t) = \int dt (-i \frac{d}{d\omega} e^{i\omega t}) f(t) = -i \frac{d}{d\omega} \tilde{f}(\omega) \ . \tag{8.63}$$

As a result, if $\hat{L} = t\partial_t$, the fourier transform of $\hat{L} f$ becomes

$$\int dt e^{i\omega t} \, t \frac{d}{dt} f(t) = \int dt (-i \frac{d}{d\omega} e^{i\omega t}) \frac{d}{dt} f(t) \tag{8.64}$$

$$= -i \frac{d}{d\omega} \int dt (-\frac{d}{dt} e^{i\omega t}) \frac{d}{dt} f(t) \tag{8.65}$$

$$= -i \frac{d}{d\omega} (-i\omega \tilde{f}(\omega)) \ . \tag{8.66}$$

This is a very technical way of thinking about it. There is a more beautiful way of thinking about it in terms of symmetries that hopefully you will learn more about in quantum mechanics. The short answer is that this is related to the fact that when the coefficients are constant, the equation is the same if we shift time by a constant $t \to t + c$ (i.e. the physics is the same at every time). There is an amazing fact that symmetries are related to conserved quantities and this symmetry means that energy is conserved and ω is proportional to the energy.

That being said, there are situations where the differential equation in ω is significantly easier to solve than it is in t. For example:

$$[\frac{d^3}{dt^3} + 5\frac{d^2}{dt^2} + 2t\frac{d}{dt} + 3t]x(t) \to [(-i\omega)^3 + 5(-i\omega^2) + 2 - i\frac{d}{d\omega}(-i\omega) - 3i\frac{d}{d\omega}]\tilde{x}(\omega) \tag{8.67}$$

so we replaced a differential equations with three derivatives with a differential equation with one. We can usually solve first order equations, so this can be a useful trick.

9 Partial Differential Equations

Dubin Chapter 3

Like ODEs, (linear) Partial Differential Equations (PDEs) are just an extension of what you know from the wave equation or anything from electrodynamics / electrostatics. However, unlike ODEs, even the simplest PDEs can become surprising complicated just because of boundary conditions. This is also something you have surely already experience in eletro-magnetism, which is just a set of first order linear differential equations with constant coefficients, but you know that even a slightly complicated charge distributions, conducting surface, etc. can make finding a solution quite challenging. Unlike ODEs, we will therefore focus our efforts on the simplest cases.

9.1 Basic Structure of a general linear PDE

The most general PDE takes the same form as an ODE, just with many variables

$$\hat{L}(\partial_t, \vec{\nabla}_{\vec{x}}, t, \vec{x})\Phi(t, \vec{x}) = h(t, \vec{x}) \tag{9.1}$$

As you know from electro-magnatism, there can also be systems of these equations that organize into some nice vector-like equations[4] so that $\hat{L}$ is itself a vector-like operator and $\Phi \to \vec{E}, \vec{B}$ and $h \to \vec{j}$, etc. However, to keep this discussion manageable, we fill focus on the case of a single PDE for a scalar function $\Phi(t, \vec{x})$.

For PDEs, the difference between initial value problems and boundary value problems is much more significant. Initial value problems are defined by imposing conditions at a single time:

$$\Phi(t = 0, \vec{x}) = q(\vec{x}) \qquad \dot{\Phi}(t = 0, \vec{x}) = p(\vec{x}) \tag{9.2}$$

The non-trivial fact is that we have to specify a whole function of $\vec{x}$, but this will rarely impact how we think about the solutions to the equation or the dynamics in general. There are even physical situations where want to impose $q(\vec{x}) = p(\vec{x}) = 0$ or $q(\vec{x}) = q_0$ and $p(\vec{x}) = p_0$, where the initial value problem is very similar to the ODE case.

In contrast, boundary value problems are significantly more complicated in more than one dimension of space[5]. For example, imagine we are solving the Laplace equation (electrostatics) in 3-dimensions where I have a conductor that is a complicated shaped surface, γ. It is easy to state the boundary condition (vanishing electric potential at the conductor)

$$\Phi(\vec{x})|_{\vec{x}\in\gamma} = 0 \ , \tag{9.3}$$

but solving this equation can be very difficult because of the shape of γ. If you have solved this problem for a point change and one conducting sphere, you are already familiar with how even a very simple surface can be fairly complicated.

[4] Notice also that I wrote the $\hat{L}$ in terms of $\nabla_{\vec{x}}$, anticipating that our equations will not distinguish x, y and z (i.e. it looks the same if we rotate out coordinates). Even this assumption need not be true, as waves inside materials can have a preferred origination. It should be clear how to generalize to these more complicated circumstances using the tools we will develop for the simplest cases.

[5] Technically, 1 space + 1 time dimension is enough to show the complication, but people often assume that boundary value problems as boundary conditions in $\vec{x}$ that are independent of t, as we will do here.

For our purposes, we will usually impose two types of boundary conditions:

$$\Phi(\vec{x},t)|_{\vec{x}\in\gamma} = q(\vec{x}\in\gamma,t) \qquad \text{Dirichlet BCs} \tag{9.4}$$

$$\hat{n}\cdot\vec{\nabla}\Phi(\vec{x},t)|_{\vec{x}\in\gamma} = p(\vec{x}\in\gamma,t) \qquad \text{Neumann BCs ,} \tag{9.5}$$

where $\hat{n}$ is the direction normal to the surface. With electromagnetism, you are also often interested in conditions on derivatives parallel to the surface, but these two will be complicated enough for us in this course.

9.2 Structure of Simple Linear PDEs

For the purpose of this course, we will focus only on cases where $\hat{L}(\partial_t, \vec{\nabla}_{\vec{x}})$ and is, at most, second order in any one of the derivatives. To make life even easier, we will assume we are solving for a single function Φ and everything is rotationally symmetric so that derivatives with respect to $\vec{x}$ must take the form $\vec{\nabla}\cdot\vec{\nabla} \equiv \vec{\nabla}^2$. With these constraints, there are essentially 4 simple classes of equations of cover a very wide range of physical applications:

$$\vec{\nabla}^2\Phi(\vec{x}) = h(\vec{x}) \qquad \text{Laplace Equation} \tag{9.6}$$

$$\left[\partial_t - \alpha\vec{\nabla}^2\right]\Phi(t,\vec{x}) = h(t,\vec{x}) \qquad \text{Heat Equation} \tag{9.7}$$

$$\left[\partial_t^2 - c_s^2\vec{\nabla}^2\right]\Phi(t,\vec{x}) = h(t,\vec{x}) \qquad \text{Wave Equation} \tag{9.8}$$

$$\left[i\hbar\partial_t + \frac{\hbar^2}{2m}\vec{\nabla}^2\right]\Phi(t,\vec{x}) = h(t,\vec{x}) \qquad \text{Schrödinger Equation} \tag{9.9}$$

We will leave the Schrödinger Equation to quantum mechanics, but there is nothing particularly different about how we solve the equation. For all four of these equations, the same tricks apply. In particular, each equation involves a $\vec{\nabla}^2\Phi$ which controls all of the tricks at our disposal for dealing with the boundary conditions.

Just like ODEs, we understand how these equations work by setting $h(t,\vec{x})$ and solving the homogenous equation. The case of $h(t,\vec{x}) \neq 0$ is solved using Green's function, which are essentially built from these solutions anyway (as we saw in the case of ODEs).

9.3 Separation of Variables for the Homogenous Equation

The basic idea that underlies all of these equations is that we can find solutions using the following ansatz:

$$\Phi(t,\vec{x}) = F(t)\times G(\vec{x}) \tag{9.10}$$

In rectangular coordinates we can further decompose $G(\vec{x}) = g_x(x)g_y(y)g_z(z)$ (this might not be the best choice depending on the boundary conditions). The reason this will work is that none of these equations have mixed t and $\vec{x}$ derivatives (or mixed x,y,z derivatives). Let's see how this

works in the heat equation (again, taking $h(t,\vec{x}) = 0$):

$$\begin{aligned}
\partial_t(F(t)G(\vec{x})) &= \alpha\vec{\nabla}^2(T(t)G(\vec{x}) \\
G(\vec{x})F'(t) &= F(t)\alpha\vec{\nabla}^2G(\vec{x}) \\
\frac{F'(t)}{\alpha F(t)} &= \frac{\vec{\nabla}^2G(\vec{x})}{G(\vec{x})} \qquad (9.11) \\
\frac{F'(t)}{\alpha F(t)} &= \frac{g_y(y)g_z(z)g_x''(x) + g_x(x)g_z(z)g_y''(y) + g_x(x)g_y(y)g_z''(z)}{g_x(x)g_y(y)g_z(z)} \\
\frac{F'(t)}{\alpha F(t)} &= \frac{g_x''(x)}{g_x(x)} + \frac{g_y''(y)}{g_y(y)} + \frac{g_z''(z)}{g_z(z)} \qquad (9.12)
\end{aligned}$$

At first sight, this might not look that remarkable as I still have what looks like a mess of an equation. To see why this equation is powerful, imagine I am trying to solve the equation:

$$f(x,p_i) = g(y,q_j) \tag{9.13}$$

for all x and y, in terms of some parameters p_i and q_j that define these two functions. Since x and y are independent variables, there is only one way to solve this equation:

$$f(x,p_i) = k = g(y,q_j) \tag{9.14}$$

where k is some constant. To see why this is true, notice that this equation is true if I just fix $y =$ constant so the the RHS of this equation is a constant. So clearly then $f(x,p_i)$ must also be a constant. Repeating the logic on the LHS, we see that both sides must be independent of x and y and are therefore just constant.

9.4 Eigenvalue problems in Rectangular Coordinates

In the above example, it is presumably clear that I am just solving one of two possible equations:

$$T'(t) = \alpha k T(t) \tag{9.15}$$

or

$$g''(x) = k_x g(x) \tag{9.16}$$

(and similarly for y and z). These are called eigenvalue problems because they are equations of the form:

$$\hat{L}f = \lambda f \ . \tag{9.17}$$

where $\hat{L}$ is a differential operator and λ is a constant. This use of words should remind you of linear algebra where the eigenvalues (λ_i) and eigenvectors ($\vec{v}_i$) of a matrix (M) were defined such that

$$M\vec{v}_i = \lambda_i\vec{v}_i \tag{9.18}$$

If you remember that $\hat{L}$ and M are just example of linear operations, then this is really just the same thing: a linear operation on a object that returns the object times a constant. (The fourier transform is itself a linear operation and in your homework you found that the Hermite functions are the eigenfunctions of this operation.)

In rectangular coordinates, you already know everything you need to solve these equations: you all know that $e^{\lambda t}$ is an eigenfunction of the derivative operator so clearly we can just write $T(t) = e^{\lambda_t t}$ and $g_i(t) = e^{\lambda_i t}$ to get

$$\lambda_t = \alpha k_t \qquad \lambda^2_{x,y,z} = k_{x,y,z} \qquad \alpha k_t = k_x + k_y + k_z \tag{9.19}$$

So clearly we can find solutions to the equation pretty easily. But, as we have emphasized, what makes PDEs hard is the boundary conditions, not finding functions that satisfy the equation itself. A related question is which of these $\lambda_{t,x,y,z}$ are we allowed to sum over before / after imposing our boundary conditions.

Imposing boundary conditions

Separation of variables and the eigenvalue problem are only useful if the boundary conditions can be imposed separately on each of the functions. E.g. we impose boundary conditions at fixed t, x, y and/or z. First let's assume this is the case and see how to solve the equation.

Suppose our boundary conditions are something like $\Phi(t, x = 0, y, z) = \Phi(t, x = L, y, z) = 0$ and similarly for y and z. We will need an initial condition as well but since, λ_t is determined in terms of $\lambda_{x,y,z}$ it makes sense to deal with the boundary conditions first.

This particular set of boundary conditions can clearly be solved by setting

$$g_{x,y,z}(x, y, z = 0) = g_{x,y,z}(x, y, z = L) = 0 \tag{9.20}$$

Clearly if I can solve these boundary conditions with a single eigenfunction, the it will be part of my solution. There is already one subtelty as this point: $k_{x,y,z} = \lambda^2_{x,y,z}$ so $\pm\lambda_{x,y,z}$ both give the same $k_{x,y,z}$. Separation of variables only works if the eigenvalues all match, but if we have more than one function with the same eigenvalue, then we can / must add them together:

$$g_x = c_1 e^{+\lambda_x x} + c_2 e^{-\lambda_x x} \ . \tag{9.21}$$

This is important because $e^{\lambda x}$ is never zero, for both real and imaginary λ_x, so we must include both solution to have a hope of solving this equation.

The next question is whether we are missing solutions where the individual eigenfunctions (with different eigenvalues) do not satisfy the boundary conditions, but there is a sum of eigenfunctions that does. Let us suppose that were true, then we would have

$$\Phi(x = 0) = 0 = \sum_i c_i T_i(t) g_{x,i}(0) g_{y,i}(y) g_{z,i}(z) \tag{9.22}$$

where $g_{x,i}(0) \neq 0$ for all i. This equation can only be solved if $T_i(t) \propto T_{j\neq i}(t)$ and similarly for $g_{y,i}(z)$ and $g_{z,i}(z)$. However, by construction at least one of eigenvalues of these three functions must be different because $\lambda_{x,i} \neq \lambda_{x,j}$ by assumption. As a result, it is impossible to cancel the non-zero contributions between functions with different eigenvalues.

This conclusion is, of course, special to this particular set of boundary conditions. Suppose instead that we imposed boundary conditions so that $\Phi(t, x, y, z) = 0$ when $x^2 + y^2 + z^2 = L^2$ (i.e. on a sphere). Clearly I can't impose the boundary conditions separately on each of the $g_{x,y,z}$. On the other hand, they are still solutions to the equation, so I must be able to think of whatever answer I get as some linear combination of these solutions. It will just be horribly complicated.

9.5 Example 1: Heat Equation in One-Dimension

As an example are going to study the equation:

$$[\partial_t - \alpha\partial_x^2]\Phi(t,x) = 0 \tag{9.23}$$

with the boundary conditions $\Phi(x=0,t) = \Phi(x=L,t) = 0$. Separation of variables tells us that the a solution takes the form:

$$\Phi(t,x) = T(t)G(x) = e^{\alpha\lambda t}[ae^{\sqrt{\lambda}x} + be^{-\sqrt{\lambda}x}] \tag{9.24}$$

for $\lambda \neq 0$. There is also two zero eigenvalue solutions:

$$\Phi(t,x) = T(t)G(x) = a + bx \tag{9.25}$$

Now let's go through the possibilities to see what the possible solutions are that satisfy our boundary conditions:

- $\lambda > 0$: We have two equations

$$a + b = 0 \qquad ae^{\sqrt{\lambda}L} + be^{-\sqrt{\lambda}L} = 0 \rightarrow a = b = 0 \tag{9.26}$$

  Solve the conditions as $x = 0$ sets $a = -b$ and so the second equationis $a(1 - e^{-2\sqrt{\lambda}L}) = 0$. For $\lambda > 0$ the only way to solve this is to set $a = -b = 0$.

- $\lambda = 0$: Like the first case we have

$$a = 0 \qquad a + bL = 0 \rightarrow a = b = 0 \tag{9.27}$$

- $\lambda < 0$: We will rewrite the square root as $\sqrt{\lambda} \equiv ik$ and we get

$$a + b = 0 \qquad [ae^{ikL} + be^{-ikL}] = 0 \tag{9.28}$$

  Now we can solve this with $a = -b$ and $k = \pi n/L$ for some integer n.

Putting this together we have

$$\Phi(t,x) = \sum_{n=1}^{\infty} b_n e^{-\alpha\pi^2 n^2 t/L^2} \sin(\pi n x/L) \tag{9.29}$$

Now we impose the initial conditions, $\Phi(t=0,x) = \Phi_0(x)$ and we will just fourier transform to find

$$b_n = \frac{2}{L}\int_0^L dx \Phi_0(x) \sin(\pi n x/L) \tag{9.30}$$

Notice that since $n > 0$, as $t \rightarrow \infty$, $\Phi(t,x) \rightarrow 0$. This is kind of obvious, as we are cooling a material at both ends, so the long time behavior should be that it reaches equilibrium.

This looks essentially like we are just solving the equation by fourier transform. Why didn't we just take the fourier transfrom from the beginning? In this case, we ended up with something that looks just like a fourier transform because we picked boundary conditions that are are

consistent with it being periodic boundary conditions since $\Phi(t, x = 0) = \Phi(t, x = L)$. So let's see what happens if we pick boundary conditions that are not periodic:

$$\Phi(t, x = 0) = 0 \qquad \Phi(t, x = L) = T_0 > 0 \tag{9.31}$$

Now we have to revisit the question of boundary conditions. Holding $\Phi(t, x = L) = T_0 > 0$ is non-trivial. For $\lambda \neq 0$, $T(t)$ depends on time and therefore there is no way maintain a boundary conditions for all time.

However, if we have a solution to the equation with $\lambda = 0$ that satisfies the boundary conditions, the answer is time independent. Now if we have a single solution that satisfies $\Phi_{\lambda=0}(t, x = L) = T_0$, we can impose $\Phi_{\lambda\neq 0}(t, x = L) = 0$. Remembering that the solution for $\lambda = 0$ was $a + bx$ we have

$$a = 0 \qquad a + bL = T_0 \ , \tag{9.32}$$

we just add this solution to our previous solution to find

$$\Phi(t, x) = \frac{x}{L} T_0 + \sum_{n=1}^{\infty} b_n e^{-\alpha \pi^2 n^2 t/L^2} \sin(\pi n x/L)] \tag{9.33}$$

where now the coefficients are

$$b_n = \frac{2}{L} \int_0^L dx (\Phi_0(x) - \frac{x}{L} T_0) \sin(\pi n x/L) \ . \tag{9.34}$$

Physically, this makes a tone of sense. $\lambda = 0$ is the equilibrium solution (i.e. $\partial_t \Phi(t, x) = 0$). So if we wait, everything should decay away and we should be left with the equilibrium solution.

As last example in one dimension, we will consider the case of Neumann boundary conditions:

$$\partial_x \Phi(x = 0, t) = \partial_x \Phi(x = L, t) = 0 \tag{9.35}$$

Before we solve the equation, we should understand the physics of this boundary condition. This equation says that the temperature at the boundary of the material isn't fixed but that there is no temperature gradient. To make sense of this equation, let us recall that the heat flow across a surface is related to the temperature gradient $\vec{q} = -k\vec{\nabla} T$. In other words, heat flows from hot to cold. So, if we set $\hat{n} \nabla T = 0$, then we are requiring that no heat flows through the boundary.

Again, the eigenfunction as the same as before, Eqn (9.24), but we need to check which solutions are allowed by the boundary conditions

- $\lambda > 0$: We have two equations

$$\sqrt{\lambda}(a - b) = 0 \qquad \sqrt{\lambda} e^{\sqrt{\lambda} L} - b e^{-\sqrt{\lambda} L} = 0 \to a = b = 0 \tag{9.36}$$

  Nothing much changed here.

- $\lambda = 0$: Like the first case we have

$$b = 0 \qquad b = 0 \to b = 0 \tag{9.37}$$

  Notice that we can have a being anything we want.

- $\lambda < 0$: We will rewrite the square root as $\sqrt{\lambda} \equiv ik$ and we get

$$ika - ikb = 0 \qquad ik[ae^{ikL} - be^{-ikL}] = 0 \tag{9.38}$$

Now we can solve this with $a = b$ and $k = \pi n/L$ for some integer n.

So now our solution is

$$\Phi(t,x) = \sum_{n=0}^{\infty} a_n e^{-\alpha\pi^2 n^2 t/L^2} \cos(\pi n x/L)] \tag{9.39}$$

where

$$a_n = \frac{2}{L}\int_0^L dx \Phi_0(x) \cos(\pi n x/L) \ . \tag{9.40}$$

for $n > 0$ and

$$a_0 = \frac{1}{L}\int_0^L dx \Phi_0(x) \ . \tag{9.41}$$

Notice now that as $t \to \infty$ we have $\Phi(t,x) \to a_0$ which is just the average temperature. We don't allow heat to flow out the ends, so the temperature just moves around to make a universe temperature but the average doesn't change.

Relation to Wave/Schrödinger Equation

It is worth noting that very little about our solutions were specific the heat equation. Notice that we spend all of our time determining the solutions to $\nabla^2 g(x) = \lambda^2 g(x)$ and imposing boundary conditions. This would have been the same problem for the Wave equation or the Schrödinger equation. Specifically, for the one-dimensional wave equation with Dirichlet boundary conditions (waves on a string) the only different is that the equation for time becomes

$$T'' = c_s^2 \lambda T \to T_n = c_1 e^{ic_s tn\pi/L} + c_2 e^{-ic_s tn\pi/L} = a_n \cos(c_s tn\pi/L) + b_n \sin(c_s tn\pi/L) \tag{9.42}$$

This is before we impose initial conditions at $t = 0$. Since we have two constants to determine, we need two conditions $\Phi(t = 0, x) = Q(x)$ and $\dot{\Phi}(t = 0, x) = P(x)$ then

$$Q(x) = \sum_n a_n \sin x\pi n/L \qquad P(x) = \sum_n b_n \sin x\pi n/L \tag{9.43}$$

So we again determine a_n and b_n by integration:

$$a_n = \frac{2}{L}\int_0^L dx Q(x) \sin x\pi n/L \qquad b_n = \frac{2}{L}\int_0^L dx P(x) \sin x\pi n/L \tag{9.44}$$

We should also notice that the discrete spacing in the allowed eigenvalues is common to all of these equations. For the wave equation, we think of this as a set of harmonics associated to a fundamental mode of oscillation. In quantum mechanics, these are energy levels. We should just remember that this appearance of discrete quantites from continuous equations is incredibly common and isn't fundamentally tied to quantum mechanics, waves or anything else.

9.6 Polar Coordinates

The simplest extension of rectangular coordinates that arises in many physical systems are polar coordinates. We already encountered this for complex variables: instead of writing $z = x + iy$ we write $z = \rho e^{i\varphi}$ so that $x = \rho\cos\varphi$ and $y = \rho\sin\varphi$.

Suppose we are solving the heat / wave equation in two dimensions but our boundary conditions are defined as $\Phi(t, x, y) = 0$ when $x^2 + y^2 = R^2$. If we could instead separate variables into ρ and ϕ this is just

$$\Phi(t, \rho = R, \varphi) = 0 \tag{9.45}$$

For this to work, we need to write the Laplacian in terms of ρ and φ, not x and y. This is easier if we remember that

$$\partial_z = \partial_x + \frac{1}{i}\partial_y \tag{9.46}$$

so that

$$\vec{\nabla}^2 = \partial_x^2 + \partial_y^2 = \partial_z\partial_{z^*} \tag{9.47}$$

Now using the chain rule

$$\begin{aligned}
\partial_z &= \frac{\partial\rho}{\partial z}\frac{\partial}{\partial\rho} + \frac{\partial\varphi}{\partial z}\frac{\partial}{\partial\varphi} && (9.48)\\
&= e^{-i\varphi}\frac{\partial}{\partial\rho} - i\frac{e^{-i\varphi}}{\rho}\frac{\partial}{\partial\varphi} && (9.49)\\
\partial_{z^*} &= \frac{\partial\rho}{\partial z^*}\frac{\partial}{\partial\rho} + \frac{\partial\varphi}{\partial z^*}\frac{\partial}{\partial\varphi} && (9.50)\\
&= e^{i\varphi}\frac{\partial}{\partial\rho} + i\frac{e^{i\varphi}}{\rho}\frac{\partial}{\partial\varphi} && (9.51)
\end{aligned}$$

so now we have

$$\begin{aligned}
\vec{\nabla}^2 &= \left(e^{-i\varphi}\frac{\partial}{\partial\rho} - i\frac{e^{-i\varphi}}{\rho}\frac{\partial}{\partial\varphi}\right)\left(e^{i\varphi}\frac{\partial}{\partial\rho} + i\frac{e^{i\varphi}}{\rho}\frac{\partial}{\partial\varphi}\right) && (9.52)\\
&= \frac{\partial^2}{\partial\rho^2} - \frac{i}{\rho^2}\frac{\partial}{\partial\varphi} + \frac{i}{\rho}\frac{\partial^2}{\partial\rho\partial\varphi} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{i}{\rho}\frac{\partial^2}{\partial\rho\partial\varphi} + i\frac{1}{\rho^2}\frac{\partial}{\partial\varphi} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} && (9.53)\\
&= \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} && (9.54)
\end{aligned}$$

You don't need to remember this derivation, but just to understand roughly where the result comes from. (You can also do it with the chain rule using $y = \rho\sin\varphi$ and $x = \rho\cos\varphi$ but it is more tedious and much less insightful.)

Armed with this expression, we can now attack problems problems new types of boundary conditions. Suppose we are solving the heat equation in two-dimensions. Using seperation of variables we have

$$\Phi(t, \rho, \varphi) = T(t)g(\rho)p(\varphi) \tag{9.55}$$

and we find

$$\frac{T'}{\alpha T} - \left[\frac{g''}{g} + \frac{g'}{\rho g} + \frac{1}{\rho^2}\frac{p''}{p}\right] = 0 \tag{9.56}$$

$$T' = \alpha\lambda T \tag{9.57}$$

$$\frac{g''}{g} + \frac{g'}{\rho g} + \frac{1}{\rho^2}\frac{p''}{p} = \lambda \tag{9.58}$$

Now we take the last equation an rewrite it as

$$\rho^2\frac{g''}{g} + \rho\frac{g'}{g} + \frac{p''}{p} = \rho^2\lambda \tag{9.59}$$

Now applying the same logic again, we have

$$\frac{p''}{p} = -n^2 \tag{9.60}$$

I have labeled it this way because of an educated guess: φ is a periodic variable, so that $p(0) = p(2\pi)$ is our boundary condition. As a result, we can see $p = e^{in\varphi}$ where n is an integer. Now we get:

$$\rho^2 g'' + \rho g' + (-\rho^2\lambda - n^2)g = 0 \tag{9.61}$$

Although we started with a PDE with only constant coefficients, we see that it has lead us to solve an ODE with non-constant coefficients. We now have two constants to keep track of as we enumerate possible solutions:

- $n = 0$, $\lambda = 0$: $g = a + b\log\rho$. Imposing the boundary conditions that $g(\rho = R) = 0$ and that $g(\rho = 0)$ is finite gives $a = b = 0$.
- $n \neq 0$, $\lambda = 0$: $g = a\rho^n + b\rho^{-n}$ so again, the boundary conditions and that it is finite at $\rho = 0$ requires $a = b = 0$.
- $\lambda > 0$: $g = aK_n(\sqrt{\lambda}\rho) + bI_n(\sqrt{\lambda}\rho)$. These functions are a generalization of the exponentials and have no zeroes. However, in the limit $\rho \to 0$, $K_n(\rho \to 0) \to \infty$ so maintaining a finite solution requires that $a = 0$. The boundary condition at $\rho = 0$ is the $bI_n(\sqrt{\lambda}R) = 0$ which can only be satisfied with $b = 0$.
- $\lambda < 0$: $g = aJ_n(k\rho) + bY_n(k\rho)$, where $k^2 = -\lambda$. $Y_n(x \to 0) \to \infty$ so we set $b = 0$. $J_n(k\rho)$ is some generalization of sin and has an infinite number of zeros x_j, so that $J_n(x_j) = 0$. Imposing boundary conditions requires that $k_{j,n} = x_{j,n}/R$.

The easiest way to understand this pattern is to notice that when $\rho \to \infty$, this equation looks like $g'' - \lambda g + \mathcal{O}(\frac{1}{\rho}) = 0$ so the solutions asymptote to $e^{\pm\sqrt{\lambda}\rho}$ or $e^{\pm ik\rho}$ depending on the sign of λ. For this reason, what λ values are allowed looks essentially the same as the exponential case, expect that the zeros of the Bessel function aren't just at πn. In addition, when $\rho \to 0$, we can essentially ignore λ and the solutions are $\rho^{\pm m}$ as $\rho \to 0$ and therefore one of the two solutions was always going to be excluded when $m \neq 0$ because it diverges as $\rho \to 0$.

Using these results, the full solution becomes

$$\Phi = \sum_{j,n} c_n e^{-\alpha x_{j,n}^2 t/L^2} J_n(x_{j,n}\rho/R) e^{in\varphi} \tag{9.62}$$

Now, we want to impose initial condition $\Phi(t=0,\rho,\varphi) = \Phi_0(\rho,\varphi)$. If we want to follow what we did before, we need some way to related Φ_0 and c_n. This is where we can use a remarkable fact:

$$\int_0^1 d\rho\rho J_n(x_{j,n}\rho) J_n(x_{j',n}\rho) = \frac{1}{2} J_{n+1}(x_{j,n})\delta_{j,j'} \tag{9.63}$$

So suppose that

$$\Phi_0(\rho,\varphi) = \sum_{j,n} c_{j,n} J_n(x_{j,n}\rho/R) e^{in\varphi} \tag{9.64}$$

then

$$2\pi \frac{R^2}{2} J_{n+1}(x_{j',n'}) c_{j',n'} = \int_0^{2\pi} d\varphi \int_0^R d\rho\rho J_{n'}(x_{j',n'}\rho/R) e^{-in'\varphi} \Phi_0(\rho,\varphi) \tag{9.65}$$

It might seem miraculous that we have this nice "orthogonality" property of the Bessel functions that allow us to determine the coefficients like this is a fourier transform. This is no a coincidence: if you learn more about Sturm-Louiville problems, you will find that for a very large number of ODE eigenvalue problems, there exists some kind of integral over two eigenfunctions that vanishes is they have different eigenvalues.

This is essentially the function version of the statement that eigevectors of a symmetric matrix with distinct eigenvalues are orthogonal. This follows from the fact that $(v_1^{\mathrm{T}} M)v_2 = \lambda_1 v_1 \cdot v_2 = v_1^{\mathrm{T}}(Mv_2) = \lambda_2 v_1 \cdot v_2$. *Since* $\lambda_1 \neq \lambda_2$ *the only way this equation can be true is if* $v_1 \cdot v_2 = 0$.

This fact is extremely useful even if you don't care about the ODE's themselves: just like the fourier transform, there are lots of reasons you might want to rewrite a function in terms of some basis of functions with nice properties. E.g. suppose you have a data set that lives on a disk of radius L. It is nice to know you can decompose this information in terms of Bessel functions and $\cos m\varphi$. We will see this again with spherical harmonics, which have a lots of applications outside of solving the eigenvalue problem for the Laplacian.

Cylindrical Coordinates

There is a natural generalization of polar coordinates to three dimensions that have cylindral symmetries, so you write $\Phi(t,\rho,\varphi,z)$, where we wrote x, y in polar form and left z unchanged. This is useful for finding standing waves inside a cylinder, for example. We will not discuss this case in detail because it just involved combining the two ideas. Clearly we have

$$\Phi(t,\rho,\varphi,z) = T(t)g(\rho)p(\varphi)Z(z)\ . \tag{9.66}$$

The only non-trivial fact is that now we have three eigenvalues, λ_t, $\lambda_{g,n}$ and λ_z that are related by $\alpha\lambda = \lambda_{g,n} + \lambda_z$.

9.7 Spherical Harmonics

Spherical harmonics are just the generalization of polar / cylindrical harmonics to cases where we impose boundary conditions on the surface of a sphere in three dimensions (harmonic functions are functions that solve the Laplace equation, so these names are just refer to solutions to the Laplace equation in different coordinate systems).

Unlike the case of polar coordinates, we will not derive the Laplace equation. It follows from the same principles, but it is mostly tedious to work out. Recall the spherical coordinates are described by the radius, r and two angles φ and θ such that

$$z = r\cos\theta \qquad x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \tag{9.67}$$

The way to remember this is that we take the vector project it onto the z-axis ($r\cos\theta$), and the different is some vector of length $r\sin\theta$ somewhere in the x, y-plane. We then decompose that vector as usual with $\cos\varphi$ and $\sin\varphi$.

Now, I will claim without proof that the Laplacian in these coordinates is

$$\nabla^2 f = \frac{1}{r^2}\partial_r(r^2\partial_r f) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta f) + \frac{1}{r^2\sin\theta^2}\partial_\varphi^2 f \tag{9.68}$$

We can recognize the ∂_φ^2 is the same as before but now has $1/\sin^2\theta$ as we would have guess from the way we decomposed the coordinates.

As before, we will separate variables and solve the eigenvalue problem: $\Phi(t, r, \theta, \varphi) = T(t)g(r)p(\theta)q(\varphi)$ so that

$$\frac{g'' + 2g'/r}{g} + \frac{p'' - p'/\tan\theta}{r^2 p} + \frac{q''}{\sin\theta^2 r^2 q} = \lambda \tag{9.69}$$

First we multiply by r^2 to get

$$\frac{r^2 g'' + 2rg'}{g} - r^2\lambda = \frac{p'' - p'/\tan\theta}{p} + \frac{q''}{\sin\theta^2 q} \tag{9.70}$$

So the LHS and RHS must be equal to some constant L.

Angular Equations

Now focusing on the angular part, we have

$$\frac{p'' - p'/\tan\theta}{p} + \frac{q''}{\sin\theta^2 q} = L \to \sin^2\theta\frac{p'' - p'/\tan\theta}{p} - L\sin^2\theta = -\frac{q''}{q} \tag{9.71}$$

We again equate this to a constant, $-m^2$ and write $q = e^{im\varphi}$ as we did in the case of polar coordinates. Now the tricky part, our equation for $p(\theta)$ is

$$\sin^2\theta\frac{p'' - p'/\tan\theta}{p} - L\sin^2\theta = m^2 \tag{9.72}$$

It is not obvious what to do with this. But it is useful to remember that the volume element in three dimensions is $r^2\sin\theta dr d\theta d\varphi \to r^2 dr d\cos\theta d\varphi$, which suggests maybe $\cos\theta$ is a more useful variable than θ itself. Using $\partial_{\cos\theta} = -\frac{1}{\sin\theta}\partial_\theta$ we have

$$\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta f) = \partial_{\cos\theta}(1 - \cos^2\theta)\partial_{\cos\theta} \tag{9.73}$$

If we define $c = \cos\theta \in [-1, 1]$ then our equation becomes

$$\partial_c(1-c^2)\partial_c p(c) - Lp(c) - \frac{m^2}{1-c^2}p(c) = 0 \tag{9.74}$$

(This equation is usually written where $x = \cos\theta$, but I am using c to remind you it is a cosine and to avoid confusion with the x, y, z coordinates).

This is another example of an equation where mathematica will just tell you the answer, but it isn't very helpful:

$$p(c) = C_1 P_\ell^m(c) + C_2 Q_\ell^m(c) \tag{9.75}$$

where $L = \ell(\ell+1)$. These functions are called the associated Legendre polynomials. If you have studied quantum mechanics, this should look pretty familiar, but note that we haven't said anything about what values of ℓ and m are allowed. In particular, because $c = \cos\theta$, this functions should be smooth at $c = \pm 1$. This constraint forces the following: (i) $C_2 = 0$, (ii) ℓ is a positive integer, and (iii) $m = -\ell, \ell+1, ..., +\ell$. To see how this works, we can just expand the $m = 0$ case around $x = -1$ to find

$$P_\ell^{m=0}(c) \approx -\frac{\sin\ell\pi}{\pi}\log(1+c) + \mathcal{O}(1+c) \tag{9.76}$$

This function will diverge Logarithmically unless $\sin\ell\pi = 0$, so we see ℓ must be an integer for this logarithmic term to vanish. Since $L = \ell(\ell+1) = (-\ell)(-\ell-1) = ((-\ell-1)+1)(-\ell-1)$ so we notice that $P_\ell(c) = P_{-\ell-1}(c)$ so the set of unique Legendre polynomials that are smooth for $c \in [-1, 1]$ have $\ell = 0, 1, 2, ...$ In contrast, if you expand $Q_\ell^{m=0}(c)$ around $x = 1$ you get

$$Q_\ell^{m=0}(c) \approx -\frac{1}{2}\log(1-c) \tag{9.77}$$

which cannot be removed for any choice of ℓ and therefore $C_2 = 0$. Making this work for all m sets the above conditions (you see similar Γ functions appear at $m \neq 0$ that you can imagine giving these conditions).

As a more practical matter, these functions can be computed using the *Rodrigues' formula*:

$$P_\ell = P_\ell^{m=0} = \frac{1}{2^\ell \ell!}\partial_c^\ell (c^2-1)^\ell \tag{9.78}$$

$$P_\ell^m(c) = \frac{(-1)^m}{2^\ell \ell!}(1-c^2)^{m/2}\partial_c^{\ell+m}(c^2-1)^\ell \tag{9.79}$$

Clearly P_ℓ is just a polynomial of order ℓ. However, it is also clearly invented to have a very nice property: let us assume $\ell' = \ell + n$ for some integer $n > 0$, then

$$\int_{-1}^{1} dc P_\ell(c) P_{\ell'}(c) = \frac{1}{2^\ell \ell!}\frac{1}{2^{\ell'}\ell'!}\int_{-1}^{1} dc \partial_c^\ell (c^2-1)^\ell \partial_c^{\ell+n}(c^2-1)^{\ell+n} \tag{9.80}$$

$$= -\frac{1}{2^\ell \ell!}\frac{1}{2^{\ell'}\ell'!}\int_{-1}^{1} dc \partial_c^{\ell+1} (c^2-1)^\ell \partial_c^{\ell+n-1}(c^2-1)^{\ell+n} \tag{9.81}$$

$$\vdots \tag{9.82}$$

$$= (-1)^n \frac{1}{2^\ell \ell!}\frac{1}{2^{\ell'}\ell'!}\int_{-1}^{1} dc [\partial_c^{2\ell+n}(c^2-1)^\ell](c^2-1)^{\ell+n} \tag{9.83}$$

$$= 0 \tag{9.84}$$

In the last line, we use the fact that we are taking $2\ell + n$ derivatives of a polynomial of order 2ℓ and must vanish. We also used the fact that $\partial^{\ell+n-q}(c^2-1)^{\ell+n} \propto (1-c^2)^q \to 0$ at $c = \pm 1$ and therefore the integration by parts doesn't pick up additional terms from evaluating the function at the end points.

The above tells us that $\int_{-1}^{1} dc P_\ell(c) P_{\ell'}(c) \propto \delta_{\ell\ell'}$ and therefore make a set of orthogonal polynomials on the domain $[-1, 1]$. This might already be familiar from the multipole expansion from electro-statics (we will review this later, in case it is not). Using the actual definition, one finds:

$$\int_{-1}^{1} dc P_\ell(c) P_{\ell'}(c) = \frac{2}{2\ell+1}\delta_{\ell,\ell'} \ . \tag{9.85}$$

You could imagine arriving at the Legrendre polynomials a much less clever way, just by building a set of polynomials interatively. Here are the first few polynomials:

$$P_0(c) = 1 \qquad P_1(c) = c \qquad P_2(c) = \frac{1}{2}(3c^2 - 1) \tag{9.86}$$

This is what you would get if you thought $P_\ell(c) = \sum_{n=0}^{\ell} d_n c^n$ and then, starting from $\ell = 0$ and increasing ℓ one at a time, fix the d_n by requiring the integral with the lower order polynomials vanish.

The second part of this formula also makes sense if you remember that $\sin\theta = (1-c^2)^{1/2}$ (remember that $\theta \in [0, \pi]$ so $\sin\theta$ is always positive. The associate Legendre polynomials are just the Legrendre polynomials ($P_\ell(\cos\theta)$) where we are switching out cosines and replacing them with sines. If you think of this a basis for possible angular dependence, this makes a lot of sense. If and when you talk about this in quantum mechanics, this will make even more sense as you can show that $m \neq 0$ comes from $m = 0$ by some kind of rotation.

In terms of the associated Legendre polynomials, $P_\ell^m(\cos\theta)$, we have the following relation

$$P_\ell^{-m}(c) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(c) \tag{9.87}$$

and the orthogonality property

$$\int_{-1}^{1} dc P_\ell^m(c) P_{\ell'}^m(c) = \frac{2(\ell+m)!}{(2\ell+1)(\ell-m)!}\delta_{\ell,\ell'} \tag{9.88}$$

With this information, we have enough to define the spherical harmonics:

$$Y_{\ell,m} = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos\theta) e^{im\varphi} \tag{9.89}$$

The normalization was chosen to so that

$$\int d\cos\theta d\varphi Y_{\ell,m} Y^*_{\ell',m'} = \delta_{\ell,\ell'}\delta_{m,m'} \tag{9.90}$$

This basically just combines known properties of the individual function $e^{im\varphi}$ and $P_\ell^m(\cos\theta)$. Notice that

$$Y^*_{\ell',m'} = (-1)^m Y_{\ell',-m'} \tag{9.91}$$

This relation is somewhat trivial because we used

$$\sqrt{\frac{(\ell-m)!}{(\ell+m)!}}P_\ell^m(c) = \frac{(\ell-(-m))!}{(\ell+(-m))!}P_\ell^{-m}(c) \tag{9.92}$$

so complex conjugation is just changing the phase and we are just relabeling everything else in terms of $-m$, but they were real functions to start.

Now recall that the spherical harmonic were defined so that

$$\vec{\nabla}^2 Y_\ell^m(\cos\theta,\varphi) = \ell(\ell+1)Y_\ell^m(\cos\theta,\varphi) \tag{9.93}$$

This is important because (a) it doesn't depend on m and (b) $\ell(\ell+1)$ is the eigenvalue that will appear in the radial equation.

Application of Spherical Harmonics

It is worth taking a moment to appreciate the value of the $Y_{\ell,m}$ before we use it to find the radial solution. This piece, like the Fourier transform, has countless applications. To name a few. The most obvious is that it will show up in all of the differential equations where spherical coordinates are useful.

- They are central to quantum mechanics, including both the periodic table and describing the result of particle collisions.
- They also play a very important role in modeling tides, as we use measurements of the earth and moon's multipole moments
- The are used to describe waves propagating through the earth (seismology), the sun (hielo-seismology) and other stars (astro-seismology).
- Understanding sources of radiation, both electromagnetic (time dependent dipole) and gravitational (time dependent quadrupole).
- Describing things in the sky or in space- the sky is the surface sphere and so if we want to describe patterns in the sky that have a characteristic scale we can decompose it into spherical harmonics. The cosmic microwave background is the a great example of this (see figure). Even when we map space in 3 dimensions, what we observe usually depends on the angles in the sky and the distance from us, so we still decompose it into spherical harmonics.
- Describing the surface of the earth : just like the fourier transform, if you need to separate signal on particular scales from noise, it helps to decompose it into a basis of functions that separate scales. This is useful for mapping things on the surface of the earth on large scales, including temperature, mass distribution, etc.
- Imaging, graphics, etc

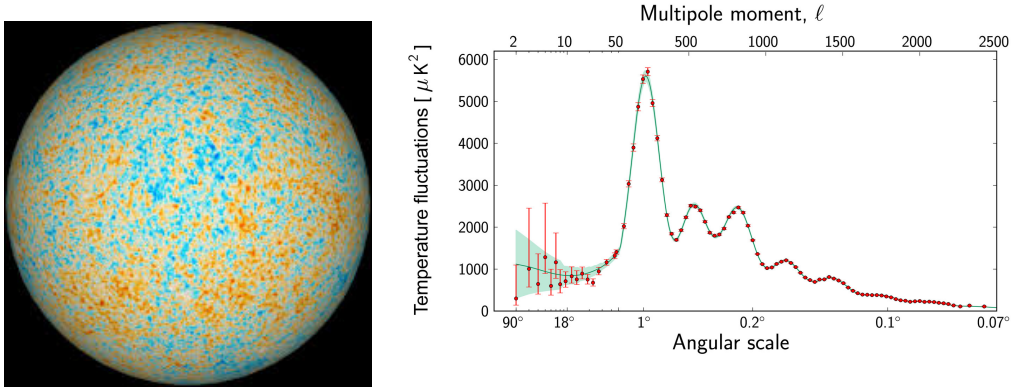

Figure 16: *Left:* Map of the cosmic microwave background on the sky. *Right:* Decomposition of the cosmic microwave background into multipoles. The pattern seen in the multipole expansion shows the properties of sound waves propagating through the universe when it was 380 000 years old.

Radial Equation

Now we are left with the radial profile, which solves the equation:

$$\frac{r^2 g'' + 2rg'}{g} - r^2\lambda = L = \ell(\ell+1) \tag{9.94}$$

Remember that m does not appear in this equation because it only appeared as a constant in separating θ and φ. Our equations is therefore:

$$r^2 g'' 2rg' + (-r^2\lambda - \ell(\ell+1))g = 0 \tag{9.95}$$

where we know that $\ell = 0, 1, 2...$ but λ can be any constant. This equation is very similar to case of polar coordinates: the only different is that we have $2rg'$ in the middle term rather than rg' in the case of polar coordinates.

Let us again work through the possible scenarios:

- $\lambda = 0$: $g = ar^\ell + br^{-\ell-1}$. Imposing the boundary conditions that $g(r = L) = 0$ and that $g(r = 0)$ is finite gives $a = b = 0$.
- $\lambda > 0$: $g = ak_\ell(\sqrt{\lambda}r) + bi_\ell\sqrt{\lambda}r)$. These are the modified *spherical bessel functions*. $a = 0$ so the function is finite. The boundary condition at $r = 0$ is the $bi_\ell(\sqrt{\lambda}L) = 0$ which can only be satisfied with $b = 0$.
- $\lambda < 0$: $g = aj_\ell(kr) + by_\ell(kr)$, where $k^2 = -\lambda$. These are the *spherical bessel functions* and imposing boundary conditions requires that $k_{i,\ell} = x_{i,\ell}/L$ where $x_{i,\ell}$ are the zeros.

Now this is just like polar coordinates but now we are using the spherical Bessel functions. But even this difference is insignificant because there is the following identity:

$$j_\ell(x) = \sqrt{\frac{\pi}{2x}} J_{\ell+\frac{1}{2}}(x) \tag{9.96}$$

We can see this by noting that if we define $g(r) = r^{-1/2}G(r)$ then our equation becomes

$$r^{-1/2}\left[r^2G''(r) + rG'(r) + (-r^2\lambda - \ell(\ell+1) - \frac{1}{4})G\right] = 0 \tag{9.97}$$

and then use $\ell(\ell+1) + 1/4 = (\ell+1/2)^2$ to recover the Bessel equation with $\ell + 1/2 = n$.

Putting this together, our solutions is

$$\Phi(t, r, \theta, \varphi) = \sum_{i,\ell,m} a_{i,\ell,m} e^{-\alpha x_{i,\ell}^2 t/L^2} j_\ell(x_{i,\ell}r) Y_{\ell,m}(\theta, \varphi) \tag{9.98}$$

We will again impose intiail condition that $\Phi(t = 0, r, \theta, \varphi) = \Phi_0(r, \theta, \varphi)$ and we will decompose

$$\Phi_0(r, \theta, \varphi) = \sum_{i,\ell,m} a_{i,\ell,m} j_\ell(x_{i,\ell}r) Y_{\ell,m}(\theta, \varphi) \tag{9.99}$$

So we again can integrate this expression (we will set $L = 1$ to reduce clutter):

$$\begin{aligned}
\int r^2 dr d\cos\theta d\varphi j_{\ell'}(x_{i',\ell'}r) Y^*_{\ell',m'} \Phi_0(r, \theta, \varphi) &= \sum_{i,\ell,m} a_{i,\ell,m} \int r^2 dr d\cos\theta d\varphi j_{\ell'}(x_{i',\ell'}r) Y^*_{\ell',m'} j_\ell(x_{i,\ell}r) Y_{\ell,m} \\
&= \sum_i a_{i,\ell',m'} \int r^2 dr j_{\ell'}(x_{i',\ell'}r) j_{\ell'}(x_{i,\ell'}r) \\
&= \sum_i a_{i,\ell',m'} \int r dr \frac{\pi}{2} J_{\ell'+\frac{1}{2}}(x_{i',\ell'}r) J_{\ell'+\frac{1}{2}}(x_{i,\ell'}r) \qquad (9.100) \\
&= a_{i',\ell',m'} \frac{\pi}{4} J_{\ell'+\frac{3}{2}}(x_{i',\ell'}r) \qquad (9.101)
\end{aligned}$$

9.8 Multipole Expansion

As you likely recall, in electrostatics or gravity the potential is determined by the Poisson equation:

$$\nabla^2\Phi(\vec{x}) = \rho(\vec{x}) \tag{9.102}$$

and the solution takes the form

$$\Phi = \int d^3x' \frac{1}{|\vec{x} - \vec{x}'|} \rho(\vec{x}') \tag{9.103}$$

It is useful to note the following:

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{\ell=0}^{\infty} \frac{r'^\ell}{r^{\ell+1}} P_\ell(\cos\theta) \tag{9.104}$$

where $|x| = r$ and $|x'| = r'$ and $\vec{x} \cdot \vec{x}' = rr'\cos\theta$. What this means is that you can decompose Φ into multipoles:

$$\Phi = \sum_\ell \Phi_\ell \tag{9.105}$$

where

$$\Phi_\ell = \frac{1}{r^{\ell+1}} \int dr' d\cos\theta d\varphi r'^{\ell+2} P_\ell(\cos\theta) \rho(r', \theta', \varphi) \tag{9.106}$$

The origin of this formula can be understood from remembering that the Green's function satisfies $\nabla^2 G(\vec{x}-\vec{x}') = 0$ for $\vec{x} \neq \vec{x}'$. This means that where $r \neq r'$ both the r and r' dependence must satisfy the Laplace equation. Now if we go back to our radial equation, we notice that we are solving the case $\lambda = 0$ for $r > r'$. The solutions were r^ℓ and $r^{-\ell-1}$ and similarly for r'. For r, we impose the condition that is goes to zero as $r \to \infty$ and for r' we impose that it is finite at $r' \to 0$, therefore the unique combination is

$$G_{\ell,m} = c_{\ell,m} \frac{r'^\ell}{r^{\ell+1}} Y_{\ell,m}(\cos\theta, \varphi) \tag{9.107}$$

now choosing coordinates so that $\vec{x}$ point only in the z-direction, we see that there can be no φ dependence so

$$G_{\ell,m} = c_\ell \frac{r'^\ell}{r^{\ell+1}} Y_{\ell,m=0}(\cos\theta) \tag{9.108}$$

This is an incredibly useful result for several reasons, but the most noteworthy is to realize that if we have an object of size a, then

$$\int dr^2 r^{\ell+2} \rho_{\ell,m=0} r \propto a^\ell \tag{9.109}$$

just from dimensional analysis (ρ is a density, so it scales like Q/a^3 or M/a^3). This means that

$$\Phi_\ell \propto \frac{a^\ell}{r^{\ell+1}} \ . \tag{9.110}$$

So if $r \gg a$, then we have a good estimate of how many terms we need to achieve a given level of accuracy. E.g. if $a/r \sim 0.1$, then if I want 0.1% accuracy (which is very good), then I will probably only need to keep $\ell = 0, 1, 2, 3$.

10 Final Summary

Theme: like Linear Algebra for Functions

A common idea that has propagated this course is that we can think of functions the same way we think about vectors and matrices. The most intuitive analogy is with a problem you know well:the motion of a particle under earth's gravity - the force acts only in the z-direction but does nothing in the x, y-directions. So you decompose your vector into x, y, z, solve for the time dependence of each one separately and add them back together. This works because (a) the problem is linear - if you write your solution as a sum, you can solve the equations individually for each term in the sum and add them back together at the end and (b) there exists a basis of vectors where the evolution in each direction is independent.

We can think of this more mathematically as well: we want to konw $\vec{x}(t)$ where forces act on this vector in some complicated way that could be described with a matrix:

$$\ddot{\vec{x}}(t) = M\vec{x}(t) \tag{10.1}$$

for some symmetric 3×3 matrix M. But we know the following about M, there exist 3 eigenvectors $\hat{x}_i$ such that

$$M\hat{x}_i = \lambda_i \hat{x}_i \ . \tag{10.2}$$

We also know that these vectors are orthogonal

$$\hat{x}_i \cdot \hat{x}_j = \delta_{ij} \tag{10.3}$$

Now we can solve the problem for any vector $\vec{x}(t)$ by writing

$$\vec{x}(t) = a_1(t)\hat{x}_1 + a_2(t)\hat{x}_2(t) + a_3(t)\hat{x}_3(t) \tag{10.4}$$

Since the equation is linear, we can solve for each one individually:

$$\ddot{a}_i(t) = \lambda_i a_i(t) \rightarrow a_i(t) = b_i e^{\sqrt{\lambda_i}t} + c_i e^{-\sqrt{\lambda_i}t} \tag{10.5}$$

Finally we impose some initial condition to determine b_i, c_i. We can figure out what the intial conditions are because we have the dot-product $\vec{x}(t=0) \cdot \hat{x}_i = a_i(t=0)$.

What we have done in this course is to extend this idea from vectors to the space of functions. We introduced the idea that a differential equation takes the form

$$\hat{L}\Phi(t, \vec{x}) = h(t, \vec{x}) \tag{10.6}$$

where $\hat{L}$ is some function of $t, \vec{x}$ and derivatives ∂_t, $\vec{\nabla}$. In a variety of circumstances, we were looking for a list of functions Φ_i that have the nice property:

$$\hat{L}\Phi_i = \lambda_i \Phi_i \tag{10.7}$$

and it also turned out that these functions have the property that they are "orthogonal"

$$\int dV \Phi_i \Phi_j = \delta_{i,j} \tag{10.8}$$

In the case of PDEs, the analogy with the force is very accurate. We had an equation

$$\partial_t \Phi = \hat{L}_x \Phi \tag{10.9}$$

These problems could all be solved the same way: find a set of eigenfunctions

$$\hat{L}\Phi_i = \lambda_i \Phi \ . \tag{10.10}$$

The solution can be written in terms of the eigenfunctions with some time dependent coefficients

$$\Phi = \sum_i a_i(t)\Phi_i \tag{10.11}$$

and we could use this "orthogonality" to determine the coefficients

$$a_i \sim \int dV \Phi_i(\vec{x})\Phi(t=0,\vec{x}) \ . \tag{10.12}$$

In the case where we were trying to understand the response to a source ($h(t,x)$), we wrote

$$\Phi = \sum_i a_i \Phi_i \tag{10.13}$$

and then can use orthogonality to write

$$\hat{L}a_i\Phi_i = \lambda_i a_i \Phi_i = \Phi_i \times \int dV \Phi_i h \tag{10.14}$$

so that

$$a_i = \frac{\int dV \Phi_i h}{\lambda_i} \tag{10.15}$$

Fourier Series

The fourier transform is the prime example we used in the course. The functions e^{-ikx} are eigenfunctions of the derivative:

$$\partial_x e^{-ikx} = (-ik)e^{-ikx} \tag{10.16}$$

so that $(-ik)$ is the eigenvalue. These functions are orthogonal

$$\frac{1}{2L}\int_{-L}^{L} dx e^{-ik_n x} e^{ik_m x} = \delta_{n,m} \tag{10.17}$$

where $k_n = \pi n/L$ so the functions obey periodic boundary conditions.

We can expand any function $f(x) = \sum_n f_n e^{-ik_n x}$ on this space by using this projection

$$f_n = \frac{1}{2L}\int_{-L}^{L} dx f(x) e^{ik_n x} \tag{10.18}$$

If you wanted to solve a differential equation on the space $[-L, L]$ with periodic boundary conditions, we could also expand the solutions as

$$\Phi(t,x) = \sum_n a_n(t) e^{-ik_n x} \tag{10.19}$$

We can extend this produced to $[-\infty, \infty]$ but now we replace the sum by the integral

$$f(x) = \int \frac{dk}{(2\pi)} e^{-ikx} \tilde{f}(k) \tag{10.20}$$

and we replace the $\delta_{n,m}$ with a new object, the Dirac δ-function

$$\int_{-\infty}^{\infty} dx e^{-ikx} e^{ik'x} = (2\pi)\delta(k - k') \tag{10.21}$$

which is defined by the property:

$$\int dx f(x)\delta(x) = f(0) \tag{10.22}$$

for any function $f(x)$. With this, we can compute the coefficients:

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx e^{ikx} f(x) \tag{10.23}$$

The fourier series / transform is very useful for solving differential equations because it is often an eigenfunction of $\hat{L}$. In particular, we should remember that

$$\partial_x \quad \leftrightarrow \quad -ik \tag{10.24}$$

$$\partial_k \quad \leftrightarrow \quad ix \tag{10.25}$$

What does this mean? You should think of the following properties

$$\partial^n x f(x) \to (-ik)^n \tilde{f}(k) \tag{10.26}$$

and

$$x^n f(x) \to -i\partial_k^n \tilde{f}(k) \tag{10.27}$$

This fact is what makes solving equation easy. For example, if I have

$$[\partial_x^{50} + \partial_y^{29}]\Phi(x, y) = x^{12} y^{12} \rho(x, y) \tag{10.28}$$

then this equation in fourier space is just

$$[(-ik_x)^{50} + (-ik_y)^{29}]\tilde{\Phi}(k_x, k_y) = (-i)^{12}\partial_{k_x}^{12}(-i)^{12}\partial_{k_y}^{12}\tilde{\rho}(k_x, k_y) \tag{10.29}$$

This is just a direct application of the fact that these are eigenfunctions of the derivatives - I shouldn't have to integrate by parts 50 times to know this is the answer.

Generalization: eigenfunctions and orthogonality

Unlike finite dimensional vector spaces, the space of functions is infinite. As a result, there are really many different "orthogonal" representations of a set of functions. For example, in this course we have seen two ways to represent functions on the space $x \in [-1, 1]$. The fourier series

represents a basis of functions with period boundary conditions on this interval. However, we also saw that we can use the Legendre polynomials to represent this function

$$f(x) = \sum_{\ell} a_\ell P_\ell(x) \tag{10.30}$$

which also obeys a type of orthogonality

$$\int_{-1}^{1} dx P_\ell(x) P_{\ell'}(x) = \frac{2}{2\ell+1}\delta_{\ell\ell'} \tag{10.31}$$

Each of these representations comes up in different physical examples, related to both the operator for which the function is an eigenfunction and the boundary conditions / geometry of the problem.

Similarly, we have seen that there are multiple ways to write the eigenfunctions of the Laplacian. For problems with retangular boundaries, the fourier series/tranform works great

$$\nabla^2 e^{i\vec{k}\cdot\vec{x}} = -\vec{k}^2 e^{i\vec{k}\cdot\vec{x}} \tag{10.32}$$

If we are instead confinded to the surface of a sphere, we found the spherical harmonics gave nice eigenfunctions

$$\nabla^2 Y_{\ell,m} = \ell(\ell+1) Y_{\ell,m} \tag{10.33}$$

Neither one is better or worse, are just more useful in different physical situations.

Finally, because these are the kinds of eigenfunctions that arise from physical situations, they tend to be very useful tools for understanding data. In particular, we can take a given data set and expand it in one of these basis of functions. E.g. a map of the sky might be more usefully expressed in spherical harmonics

$$C(\theta,\phi) = \sum_{\ell,m} a_{\ell,m} Y_{\ell,m}(\theta,\varphi) \tag{10.34}$$

One reason to do this is that our signal and our noise may behave very differently in this basis, allowing us to isolate a signal of interest. The fact that this works still has its origins in physics: the signals we are interested in usually looks nice in one of these basis of functions because eigenfunctions of differential operators are the kinds of things that come out of almost any physical system.

11 Bonus Topic: The Discrete Fourier Transfrom

Dubin: Section 2.3.5

So far we have been focused on the fourier transform as a way to understand differential equations using algebra. However, this misses the very large number of situations where the Fourier transform is applied to make sense of data. Data is collected as number of discrete measurements and thus we need to extend our ideas from functions of continuous parameters to a discrete number of points.

11.1 Big-Picture

So far, we have seen a number of situations where we have a set of "orthogonal" functions so that

$$\int dx f_n(x) f_m(x) = 0 \tag{11.1}$$

when $n \neq m$. E.g.

$$\int_{-\pi}^{\pi} dx e^{imx} e^{inx} = 2\pi\delta_{-n,m} \tag{11.2}$$

Now imagine that we have some data $T(x) = S(x) + N(x)$ where $S(x)$ is the signal we are interested in and $N(x)$ is some noise that is obscuring our signal. Of course, the problem is that we only measure the total, $T(x)$. You might think that if $|S(x)| < N(x)$ on average, then there is no hope of extracting the signal. Fortunately, there are lots of situations where you know enough about $N(x)$ to do much better. For example, image that you knew that

$$\int dx S(x) f_n(x) = S_0\delta_{n,p} \qquad \int dx N(x) f_n(x) = \frac{N_0}{n} \tag{11.3}$$

for some $p \gg 1$, S_0 and N_0. Even if $N_0 \gg S_0$, to extract our signal using $f_{n=p}$, we only need that our signal stands out above the noise for that particular function, which only requires $S_0 p > N_0$. If our signal is localized to a small number of these functions while the noise is spread out over many functions, we have use the projection onto this basis to isolate our signal and throw away our noise.

This result is a fairly simple / obvious parallel with vectors: suppose our single and noise are both vectors in an N-dimensional space so that $\vec{F} = \vec{S} + \vec{N}$. You might worry that if $|\vec{S}| \ll |\vec{N}|$ you wouldn't be able to measure $\vec{S}$. However, if we knew that $\vec{S}$ was mostly pointing in the direction of one basis vector, lets say that $\hat{x}$ then it doesn't matter what the length of $\vec{N}$, only the size of $\vec{N} \cdot \hat{x}$ matters.

This is what we intend to do with the fourier transform. We often know that our signal is some one or a few pure frequencies, and our noise is pretty much independent of frequency. In some cases, there might also be some pure frequency we want to get ride of (like a 60 Hz hum from the lights) that we can also isolate in this way.

11.2 Time-Series Analysis

The challenge of real-world data is given as some list of data points and not a continuous function. We will focus here on the case of making a measurement of a single quantity at a number of

different times. The ideas can be easily extended to multiple dimensions (e.g. images involve a value on a array of points (x_i, y_i) instead of a list).

To make life manageable, let us assume that we took measurements at N points, $\{t_0, ..., t_{N-1}\}$ and produced a list of N measurements $\{f_0, .., f_{N-1}\}$. We will define the discrete fourier transform of this to be

$$F_m = \sum_{n=0}^{N-1} f_n e^{i2\pi mn/N} \tag{11.4}$$

where $m = \{0, ..N-1\}$ and you can get back the original data using

$$f_n = \frac{1}{N} \sum_{m=0}^{N-1} F_m e^{-i2\pi mn/N} \tag{11.5}$$

A priori, it is not obvious that this procedure inverts the discrete fourier transform. In order to see this, we also need to note a small feature about our sum. Suppose we want to calculate

$$\sum_{n=0}^{N-1} (e^{-i2\pi/N})^{nm} = \sum_{n=0}^{N-1} (x_m)^n \qquad x_m = e^{-i2\pi m/N} \tag{11.6}$$

First if $m \neq 0$, then we have

$$\sum_{n=0}^{N-1} (x_m)^n = \frac{1 - x_m^N}{1 - x_m} = 0 \tag{11.7}$$

where we noticed that $x_m = e^{-i2\pi Nm/N} = 1$. In contrast, if $m = 0$, we have $\sum_{n=0}^{N-1}(1)^n = N$.

There are some familiar aspects of the DFT compared to the Fourier Series, but also some funny ones.

- Notice that we never used the list $\{t_0, ..., t_{N-1}\}$ anywhere! We are just defining a discrete operation that lives on the N-points. What this looks like is a fourier tranform on the domain $[-N/2, N/2]$.

- The list of "frequencies" is just the integers $[0, N-1]$ but we usually have frequencies at $\omega = -\infty, .. -1, 0, 1, ..\infty$. Furthermore we had the relationship $\tilde{f}(-\omega) = f^*(\omega)$. Where did the negative frequencies go? Notice that this expression is unchanged when we shift $m \to m \pm N$. So imagine that N is odd and we we make new set of labels $\tilde{m} = \{-(N-1)/2, .., -1, 0, 1, ..(N-1/)/2\}$. Then this is exactly the same as $m = \tilde{m} + N$ but now we see that $m = N-1$ is the same as $\tilde{m} = -1$ and $\tilde{m} = -(N-1)/2$ is the same as $m = (N+1)/2$.

- We see that there is something very funny happening at $m = N/2$. This is what we call the *Nyquist frequency* and it is the largest frequency that we can possibly find in our data. If we were to create data with larger frequencies $\omega > \pi$ (think $\omega_m = 2\pi m/N$), then it would just look like one of the lower frequencies. The simplest example is if we have the function

  $$f(t) = e^{i2\pi t} \tag{11.8}$$

  and then we sample it at $t = 0, 1, 2, 3, ..$ the we would just get the data $\{1, 1, 1, 1, 1, 1...\}$.

- If we want to think of this as a fourier transform on the time domain $t \in [t_{\min}, t_{\max}]$. For this domain, we know that the list of frequencies would be $k_n = n2\pi/|t_{\max} - t_{\min}|$ (this is just our formula from the fourier series with $x \in [-L, L]$ and our frequencies were $k_n = \pi n/L$). We can relate these two just by rescaling $\omega_m = 2\pi m/|t_{\max} - t_{\min}|$. We can then write $t_n = (n+1)|t_{\max} - t_{\min}|/N$ and $\omega_m = 2\pi m/|t_{\max} - t_{\min}|$ so that

$$F_{\omega_m} = e^{i\omega_m t_{\min}} \sum_{n=0}^{N-1} f_n e^{i\omega_m t_n} \tag{11.9}$$

- In terms of these rescaled frequencies, the *Nyquist frequency* is given by

$$\omega_{\text{Nyquist}} = \frac{2\pi(N/2)}{|t_{\max} - t_{\min}|} = \frac{\pi}{\Delta t} \tag{11.10}$$

  where $\Delta t = |t_{\max} - t_{\min}|/N$ is the spacing in time between the measurements. This result is related to the Nyquist-Shannon sampling theorem, which basically says if you want to measure a single of frequency ω then you need to sample it at a rate $\Delta t \leq \pi/\omega$.

- The information contained in frequencies $\omega > \omega_{\text{Nyquist}}$ are not lost but instead just show up as lower frequencies, related by a shift by $2\pi N$. E.g. the frequencies $\omega = 0, 2\omega_{\text{Nyquist}}, 4\omega_{\text{Nyquist}}, \ldots$ all show up as $\omega = 0$ in the discrete fourier transform. This fact that high frequencies you don't sample properly (by the sampling theorem criteria) show up in lower frequencies in your discrete fourier transform is known as *aliasing*.

Relation to Matrices

Somewhat surprisingly, the discrete Fourier can be treated a matrix multiplication of an N-dimentional vector. In paricular, our data f_n forms a vector and the Fourier transform is

$$\vec{F} = \mathbf{M}\vec{f} \tag{11.11}$$

where

$$M_{nm} = \omega^{n\times m} \qquad \omega_N = e^{-i2\pi/N} \ . \tag{11.12}$$

The inverse of this matrix implements the fourier transform

$$M^{-1} = \frac{1}{N} M^* \tag{11.13}$$

Interestingly, if we use a different normalization

$$\vec{F} = \mathbf{m}\vec{f} \tag{11.14}$$

with $m = \frac{1}{\sqrt{N}} M$ then this is a unitary operation, as $f^{-1} = f^*$.

11.3 Summary

The DFT is takes a list of N data points and returns the fourier transform of that data in terms of N complex numbers $\vec{F}$. If we want to think of these as frequencies associated with a conventional fourier transform, then a given F_m is associate with a frequency

$$\omega_m = \frac{2\pi m}{|t_{\max} - t_{\min}|} = m\Delta\omega \tag{11.15}$$

where $\Delta\omega = 2\pi/|t_{\max} - t_{\min}|$. The frequency that we associate to the data is actually capture physical frequencies that are related by $m + N \times q$ for any integer q. This corresponds to shifts by $2\omega_{\text{Nyquist}}$. Therefore, a given ω_m includes

$$\omega_m, \omega_m \pm 2\omega_{\text{Nyquist}}, \omega_m \pm 4\omega_{\text{Nyquist}}, ... \rightarrow \omega_m^{\text{DFT}} \tag{11.16}$$

For real data, the condition $F(\omega)* = F(-\omega)$ must still hold. It is useful to use the freedom to shift $\omega_m \rightarrow \omega_m - 2\omega_{\text{Nyquist}}$ to interpret $m > N/2$ as negative frequencies. Therefore, listing the frequencies from $m = 0$ to $m = N - 1$ we have

$$0, \Delta\omega, 2\Delta\omega, ..., \omega_{\text{Nyquist}} - \Delta\omega, \omega_{\text{Nyquist}}, -\omega_{\text{Nyquist}}, -\omega_{\text{Nyquist}} + \Delta\omega, .., -\Delta\omega \tag{11.17}$$

As a result, our condition is that $F_m^* = F_{N-m}$.

aside: some fourier transform software has a function that will shift the F_m and ω_m so that that are list in sequential order, namely $-\omega_{\text{Nyquist}}, ..0, .., +\omega_{\text{Nyquist}}$. This is a fairly simple procedure (it is just reordering a list) and you could write your own function to do this, of course.

Example

Suppose I have a data stream that composed of two pieces: a signal $s_i = a_0 \cos(\omega_0 t_i)$ and random noise n_i = random number. A given data point is then $d_i = s_i + n_i$. A very common situation is that each n_i is drawn independently from a gaussian distribution with mean $\mu = 0$ and some variance σ.

Very naively, you might think that the condition for being able to recover the signal is that its amplitude is larger than the noise, e.g. $a_0 > \sigma$. Fortunately, this is overly pesimistic. If we look at Figure 17, we see that I put in a signal approximately 3 times smaller than the noise (you can't really see it in the time series data) but when I take the fourier transform, I get a huge spike at the frequency of my signal at $\omega = 2.5$.

What about the fourier transform made this possible? Since the fourier transform can be perform on each piece seperately, let's start with the signal:

$$F_{\omega_m=\omega_0,\text{signal}} = \sum_{n=0}^{N-1} \frac{1}{2}[e^{i\omega_0 t_n} + e^{-\omega_0 t_n}]e^{i\omega_m t_n} \tag{11.18}$$

$$= \frac{1}{2}\sum_{n=0}^{N-1}[e^{i2\omega_0 t_n} + 1] \tag{11.19}$$

$$= \frac{N}{2} \tag{11.20}$$

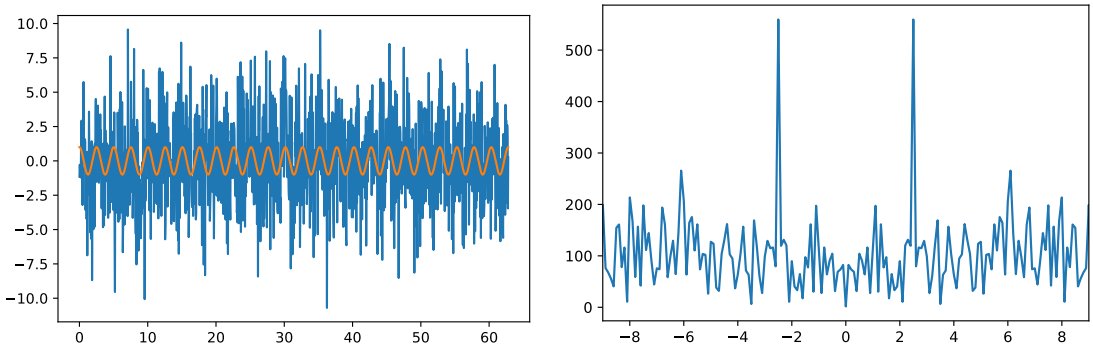

Figure 17: *Left:* Data (blue) and signal burried in the data (orange). *Right:* The discrete fourier transform of data correctly identifying the signal of $\cos(\omega t)$ with frequency $\omega = 2.5$.

where we used that the $\sum_{n=0}^{N-1} e^{i2\omega_0 t_n} = 0$ as we found earlier.

Now let's compare this to the noise itself. Since the noise is random, we want to compute some kind over average. Since $\langle n_i \rangle = 0$ (i.e. it has a mean value of 0), just averaging the equation is the wrong question. In particular, what we care about average absolute value of the fourier coefficients (e.g. how big they are) not the sign. So we should compute

$$\langle |F_{\omega_m,\text{noise}}|^2 \rangle = \langle \sum_{i,j=0}^{N-1} n_i n_j e^{i\omega_m t_i} e^{-i\omega_m t_j} \rangle \tag{11.21}$$

But now we use the fact that n_i and n_j are *independent* random variables so $\langle n_i n_j \rangle = \sigma^2 \delta_{ij}$. I.e. the average the noise at two separate points is zero so that only non-zero average is the square a single point (I used the fact that the noise if drawn from a Gaussian to say that $\langle n_i^2 \rangle = \sigma^2$. Using this result, we can eliminate one of the sums and we have

$$\langle |F_{\omega_m,\text{noise}}|^2 \rangle = \sum_{i=0}^{N-1} \langle n_i^2 \rangle e^{i\omega_m t_i} e^{-i\omega_m t_i} = \sigma^2 \sum_{i=0}^{N-1} = N\sigma^2 \tag{11.22}$$

This is the square of the absolute value, so if we take a square root:

$$\sqrt{\langle |F_{\omega_m,\text{noise}}|^2 \rangle} = \sqrt{N}\sigma \tag{11.23}$$

This is exactly what we see in our example. We have approximate 1200 data points: the signal is about 600 but the typical size of the noise at other frequencies is about $90 = 3 \times \sqrt{N}$ (remember the noise was 3 times the signal in the original data stream).

This is what makes the fourier transform such a useful tool when looking at data: if you know you are looking for signals at specific frequencies, we get a $\sqrt{N}$ enhancement of our signal over our noise as we increase the number of data points.

11.4 Usefulness of the DFT

This example showed that the idea of transforming your data from the basis in which you measured it (time series) to the basis where your signal lives (fourier) is extremely powerful. This

strategy works much more generally, regardless of whether your signal is a pure frequency: you can think of your signal as a vector $\hat{s}$ and you want to project your data onto the signal direction. Noise is random and thus will mostly not point along the signal direction leading to some kind of enhancement.

However, the Fourier transform is special amoung these types of transforms for a number of reasons:

- Harmonic oscillators and waves are very common, so it turns out lots of signal are pure frequencies. We also engineer lots of things to use frequencies. We communicate primarily through light and sound which are both examples, but we also use a lot of AC circuits in electronics instead of DC.

- The fast fourier tranforms (FFT) is an incredibly powerful algorithm. If you just use our formula, for each frequency you have to sum N points. So to get all N frequencies, you have to do N^2 operations. The FFT finds all of the numbers in just $N \log N$ operations! Imagine you have a data set where $N = 10^6$, then the FFT is $10^6/Log(10^6) = 7.2 \times 10^4$ times faster. To put that in context, if the FFT worked in 0.1 seconds, then the brute force algorithm would take 2 hrs. Similarly efficient algorithms don't exist for a lot of other applications.

- Noise can sometimes depend strongly on frequency: e.g. vibrations of the earth that rattle experiments are mostly low frequency waves. Even if our signal doesn't look like a pure frequency, we can fourier transform and project-out the noise by cutting the low or high frequencies, for example. In other words, even if our signal is not localized in the frequency, the noise can be and we can best avoid it in frequency space.

- Lots of physical effects look like convolutions. E.g. if you are looking at a start through a telescope, its position on the image is blurred by the resolution of the mirror so that you see

$$\text{obs}(x) = \int dx' b(x - x')\text{signal}(x') \tag{11.24}$$

  If you know what this blurring effect looks like, we can actually remove it pretty easily:

$$\tilde{o}(k) = \tilde{b}(k)\tilde{s}(k) \to \tilde{s}(k) = \tilde{o}(k)/\tilde{B}(k) \ . \tag{11.25}$$

  We have even fourier transform to get a nicer image with no (less) blurring. I.e.

$$\text{obs}_{\text{corrected}}(x) = \frac{1}{N}\sum_n e^{-ik_n x}\frac{O(k_n)}{B(k_n)} \tag{11.26}$$

  This is one of many examples where it is easy to "undue" effects by going to Fourier space and dividing out by the fourier transform of the thing you want to remove.