Indian Institute of Science Education and Research, Mohali

Department of Physical Science

Computational Physics II Project Report

Monit Sharma

Supervisor: Dr. Rajeev Kapri

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the Computational Physics II coursework

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Abstract

This study presents an analysis of a one-Dimensional (1D) time-dependent wave equation from a vibrating string. We consider the transverse displacement of a plucked string and the subsequent vibration motion. In the present study, we have solved the wave equation for a vibrating string using the finite difference method and analyzed the waveforms for different values of the string variables. The results show that the amplitude (pitch or quality) of the wave (sound) vary significantly with tension in the string, length of the string, the linear density of the string and also on the material of the soundboard. The approximate solution is representative; if the step width; ∂x and ∂t are small, that is < 0.5.

Contents

Chapter 1

Introduction

Ron (2002) defines a wave as "A distortion in a material or medium where the individual parts of the material only cycle back-and-forth or up-and-down, but the wave itself moves through the material". Waves exist widely in nature, such as electromagnetic waves and in many other forms of mechanical waves. Waves carry energy. The energy carried by a wave is evident in many ways.

Seismic waves carry an enormous amount of energy that shakes the earth resulting in earthquakes. Electromagnetic waves carry information in many forms, enabling communication through the internet, satellites, optical fibres, and radios. The energy from microwaves is converted into thermal energy in microwave ovens. This study presents a mathematical analysis of mechanical waves from a guitar string. Alan (2000) noted that "electromagnetic, microwaves between the earth and communication satellites, light waves in optical fibres are some of how wave energy is utilized". We can add that music is another way of utilizing wave energy.

The string is a material made of several threads twisted together. In this study, a string means a wire, a thread, nylon or any other synthetic material that is thin compared to its length, stretchable between two points. A vibrating string is just a model of many objects that vibrates in nature.

Most vibrations result in a wave motion. As already noted, waves carry energy that can be controlled for human benefits. The study of waves helps to model many valuable things in everyday life, including musical instruments, engineering devices for weather forecasts, tsunami and earthquake detection devices, and virtual communication devices.

A vibrating string presents a better and an initial point to the study of waves because the variables can easily be manipulated. The vibration of a string is time and space-dependent, a typical example of multidimensional systems. Lutz and Rudolf (2000) noted that "Multidimensional physical phenomena depending on time and space are commonly described by Partial Differential Equations (PDEs). Technical application of PDEs include electro-magnetic, optics, acoustics, heat and mass transfer.

1.1 Problem statement

Consider a string of length L and density $\rho(x)$ per unit length, tied down at both ends, and under tension $T(x)$. Please assume that the relative displacement of the string from its rest position $y(x,t)/L$ is small and that the slope $\frac{\partial y}{\partial x}$ is also tiny.

Realistic cables on bridges may be thicker near the ends in order to support the additional weight of the cable and other elements. Accordingly, the wave equation should incorporate variable density and correspondingly tension.

Extend the wave equation algorithm so that it is now appropriate to including a $T(x)$ and a $\rho(x)$.

Assume that the string's tension and density vary as $\rho(x) = \rho_0 e^{\alpha x}$, $T(x) = T_0 e^{\alpha x}$, and explore the effect of using wave equation algorithm in your simulation.

Use finite difference scheme to obtain the numerical solution of the above equation. Take $l = 1m$, $\alpha = 0.5$, $T_0 = 40N$, and $\rho_0 = 0.01$ kg/m.

1.2 Aims and objectives

Aims: To simulate the wave equation while developing the understanding of the Finite-Difference method and solving for a string of variable tension and density. It is also developing the understanding and skill for various numerical simulations in programming languages. (Eg MATLAB and Python)

Objectives: To present an analysis of a one-Dimensional (1D) time-dependent wave equation from a vibrating string using MATLAB and Python and the involving the method of finite-differences.

Chapter 2

Literature Review

2.1 Wave Equation for a Vibrating String

In the present research, the word 'string' is not used in the sense of 'string matching' algorithm as an essential means for searching biological sequence database as Almazroi (2011), Al-mazroi and Rashid (2011). Consider a single guitar string plucked from the centre, and $u(x,t)$ is the function describing a point along the string.

T_1 and T_2 are tangential tensions at points x and $x + \partial x$ respectively. Let ρ be mass per unit length of the string (linear density). The linear density is constant throughout the string, Alan (2000). As the wave propagates from point x to $x + \partial x$, the horizontal component of the tension remains a constant T.

This means

$$T_1 \cos(\theta) = T_2 \cos(\phi) = T$$

Applying Newton's second law to the vertical motion of the string, which gives us:

$$T_1 \sin(\theta) = T_2 \sin(\phi) = \rho \partial x \frac{\partial^2 y}{\partial t^2}$$

Substituing the first equation into second one,

$$T_1 = \frac{T}{\cos\theta}, T_2 = \frac{T}{\cos\phi}$$

$$T \tan(\phi) - T \tan(\theta) = \rho \partial x \frac{\partial^2 y}{\partial t^2}$$

Using elementary calculus:

$$\frac{\delta y}{\delta x} = \tan(\theta)$$

Substituing this in above equation gives :

$$T \left[\left(\frac{\delta y}{\delta x} \right)_{x+\partial x} - \left(\frac{\delta y}{\delta x} \right)_{x} \right] = \rho \partial x \frac{\partial^2 y}{\partial t^2}$$

This equation can also be written as

$$\frac{1}{\partial x} \left[\left(\frac{\delta y}{\delta x} \right)_{x+\partial x} - \left(\frac{\delta y}{\delta x} \right)_{x} \right] = \frac{\rho}{T} \partial x \frac{\partial^2 y}{\partial t^2}$$

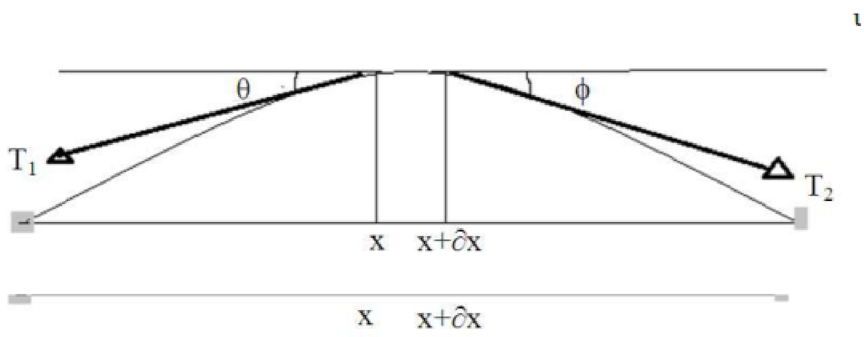


Figure 2.1: sketch showing a plucked string

Thus:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho \partial^2 y}{T \partial t^2}$$

introducing the constant $c^2 = \frac{T}{\rho}$.

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{c^2 \partial t^2}$$

This equation is the classical wave equation in one-dimension, $\frac{\partial^2 y}{\partial x^2}$ represents the second partial derivative of the wave function concerning the displacement; $\frac{\partial^2 y}{\partial t^2}$ represents the second partial of the wave function concerning time; t. The constant c represents the wave speed, and it is dependent on the tension in the string and the linear density of the string.

2.2 Wave Equation for variable density and variable tension

Realistic cables on bridges may be thicker near the ends to support the additional weight of the cable and other elements. Accordingly, the wave equation should incorporate variable density and correspondingly tension.

Realistic cables on bridges may be thicker near the ends to support the additional weight of the cable and other elements. Accordingly, the wave equation should incorporate variable density and correspondingly tension. The application of Newton's laws to leads to the wave equation:

$$\frac{dT(x)}{dx}\frac{\partial y(x,t)}{\partial x} + T(x)\frac{\partial^2 y(x,t)}{\partial x^2} = \rho(x)\frac{\partial^2 y(x,t)}{\partial t^2}$$

We can obtain our standard wave equation from above by just $c = \sqrt{T/\rho}$

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

Extending our wave equation so that it include a $T(x)$ and a $\rho(x)$, where the string's density and tension vary as:

$$\rho(x) = \rho_0 e^{\alpha x}, T(x) = T_0 e^{\alpha x}$$

Substituting them in the wave equation, we get:

$$\frac{\partial^2 y(x,t)}{\partial x^2} + \alpha\frac{T_0}{\rho_0}\frac{\partial y(x,t)}{\partial x} = \frac{T_0}{\rho_0}\frac{\partial^2 y(x,t)}{\partial t^2}$$

2.3 Boundary Conditions

Typically, we impose boundary conditions of one of the following three forms:

- *Controlled end points*: When the ends of the string are specified, we use *Dirichlet* boundary conditions of the form

$$y(0,t) = g_1(t),\ y(L,t) = g_2(t),\ t > 0$$

- *Force specified on the boundaries*: The vertical forces on the string at the endpoints are given by $Ty_x(0,t)$ and $Ty_x(L,t)$, where T is the tension in the string. By specifying these forces, we obtain *Neumann* boundary conditions. For example, if the ends of the string are allowed to slide vertically on frictionless sleeves, the boundary conditions become:

$$y_x(0,t) = 0,\ y_x(L,t) = 0,\ t > 0$$

Here $y_x = \frac{\partial y}{\partial x}$

Combinations of different boundary conditions are possible. For example, when modeling the longitudinal vibration in a spring with the end at $x = 0$ fastened and the end at $x = L$ free, the boundary conditions are

$$y(0,t) = 0, y_x(L,t) = 0, t > 0.$$

The boundary condition are

$$y(0,t) = y(L,t) = 0$$

At $t = 0$, the string is plucked at its centre. This can be modelled with the mathematical function:

$$y(x, t = 0) = \begin{cases} 1.25x/l & \text{for } x \leq 0.5l, \\ 5.0(1 - x/l) & \text{for } x > 0.5l \end{cases}$$

Chapter 3

Methodology

Partial Differential Equations (PDEs) can be solved numerically using the finite difference method. We write Python and MatLab programs for the finite difference method to generate the numerical solution to the One Dimensional (1D) wave equation.

3.1 Finite Difference Method

The finite difference method (FDM) works by replacing the region over which the independent variables in the PDE are defined by a finite grid (also called a mesh) of points at which the dependent variable is approximated.

3.1.1 Taylor's Theorem

The partial derivatives in the PDE at each grid point are approximated from neighbouring values by using Taylor's theorem.

Let $y(x)$ have n continous derivatives over the interval (a, b) . Then for $a < x_0, x_0 + h < b$

$$y(x_0 + h) = y(x) + hy_x(x_0) + h^2 \frac{y_{xx}(x_0)}{2!} + ... + O(h^n)$$

Where,

- $y_x = \frac{dy}{dx}, y_{xx} = \frac{d^2y}{dx^2}, ..$
- $y_x(x_0)$ is the derivative of y with respect to x evaluated at $x = x_0$.
- $O(h^n)$ is an unknown error term defined in Appendix A.

The usual interpretation of Taylor's theorem says that if we know the value of y and the values of its derivatives at point x_0, we can write down the equation for its value at the (nearby) point $x_0 + h$. This expression contains an unknown quantity written in as $O(h^n)$ and pronounced 'order h to the n'. If we discard the term $O(h^n)$ (i.e. truncate the right-hand side, we get an approximation to $U(x_0 + h)$). The error in this approximation is $O(h^n)$

3.1.2 Taylor's Theorem Applied to the Finite Difference Method (FDM)

In the FDM, we know the y values at the grid points, and we want to replace partial derivatives in the PDE we are solving by approximations at these grid points. We do this by interpreting Taylor's equation in another way. In the FDM, both x_0 and $x_0 + h$ are grid points and $y(x_0)$

and $y(x_0+h)$ are known. This allows us to rearrange Taylor's equation to get so-called Finite Difference (FD) approximations to derivatives with $O(h^n)$ errors. Appendix A explains the meaning of $O(h^n)$ notation.

3.1.3 Simple Finite Difference Approximation to a Derivative

Truncating the above equation after the first derivative gives:

$$y(x_0+h) = y(x_0) + hy_x(x_0) + O(h^2)$$

Rearranging gives us:

$$\begin{aligned} y_x(x_0) &= \frac{y(x_0+h)-y(x_0)}{h} + \frac{O(h^2)}{h} \\ &= \frac{y(x_0+h)-y(x_0)}{h} + O(h) \end{aligned}$$

Neglecting the $O(h)$ term gives,

$$y_x(x_0) = \frac{y(x_0+h)-y(x_0)}{h}$$

.

The above equation is called a first order FD approximation to $y_x(x_0)$ since the approximation error $= O(h)$ which depends on the first power of h. This approximation is called a forward FD approximation since we start at x_0 and step forwards to the point x_0+h. h is called the step size $(h > 0)$.

3.1.4 Building the Finite Difference Toolkit

We now construct standard FD approximations to common partial derivatives. For simplicity, we suppose that y is a function of only two variables, t and x. We will approximate the partial derivatives of y concerning x. As t is held constant, y is effectively a function of the single variable x, so we can use Taylor's formula where the ordinary derivative terms are now partial derivatives, and the arguments are (t,x) instead of x. Finally, we will replace the step size h by Δx (to indicate a change in x) so that our equation becomes:

$$y(t,x_0+\Delta x) = y(t,x_0) + \Delta x y_x(t,x_0) + \frac{\Delta x^2}{2!}y_{xx}(t,x_0) + ... + O(\Delta x^n)$$

Truncating the above equation gives,

$$y(t,x_0+\Delta x) = y(t,x_0) + \Delta x y_x(t,x_0) + O(\Delta x^2)$$

Now, we derive some FD approximation to the partial derivatives.

$$y_x(t,x_0) = \frac{y(t,x_0+\Delta x)-y(t,x_0)}{\Delta x} - \frac{O(\Delta x^2)}{\Delta x}$$

$$y_x(t,x_0) = \frac{y(t,x_0+\Delta x)-y(t,x_0)}{\Delta x} - O(\Delta x)$$

In numerical schemes for solving PDE's we are restricted to a grid of discrete x values and discrete t levels. We will assume constant grid spacing , Δx in x so that $x_{i+1} = x_i + \Delta x$. Evaluating the equation now:

$$y_x(t_n, x_i) = \frac{y(t_n, x_{i-1}) - y(t_n, x_i)}{\Delta x} - O(\Delta x)$$

We will use the common subscript/superscript notation:

$$y_i^n = y(t_n, x_i)$$

, so that dropping the $O(\Delta x)$ error term , becomes,

$$y_x(t_n, x_i) = \frac{y_{i+1}^n - y_i^n}{\Delta x}$$

This is the first-order forward difference approximation to $y_x(t_n, x_i)$

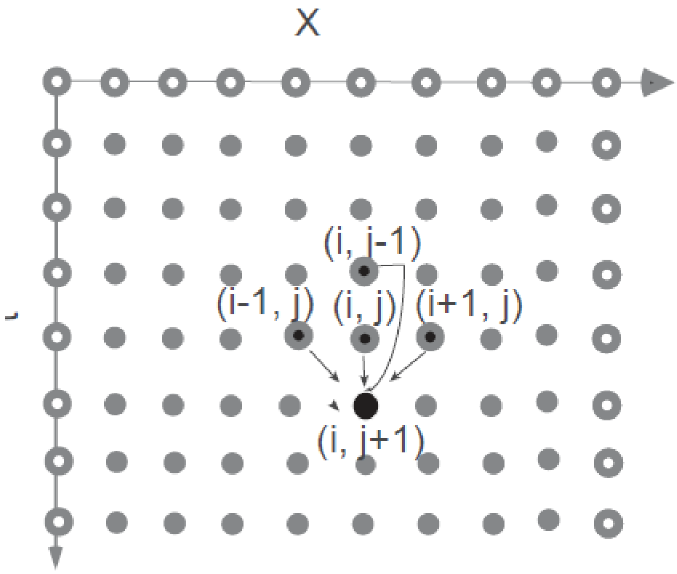


Figure 3.1: Schematic of the algorithm used to solve the wave equation. Four sites (black centers) are used to advance the solution a single time step ahead (black circle). The boundary and initial conditions are indicated by the white-centered dots.

Our first two FD approximations are first order in x but we can increase the order (and make the approximation more accurate) by taking more terms in the Taylor series. Truncating previous to $O(\Delta x^3)$, then replacing Δx by$-\Delta x$ and subtracting this new expression from the previous and evaluating at (t_n, x_i) gives, after some algebra,

$$y_x(t_n, x_i) = \frac{y_{i+1}^n - y_{i-1}^n}{2\Delta x}$$

This is called second-order central difference FD approximation.

Many PDEs of interest contain second-order (and higher) partial derivatives, so we need to derive their approximations. We will restrict our attention to second-order unmixed partial derivatives.

$$y_{xx}(t_n, x_i) = \frac{y_{i+1}^n - 2y_i^n + y_{i-1}^n}{\Delta x^2}$$

is the second order symmetric difference FD approximation to $y_{xx}(t_n, x_i)$ Similarly for the time derivative:

$$y_{tt}(t_n, x_i) = \frac{y_i^{n+1} - 2y_i^n + y_i^n - 1}{\Delta t^2}$$

3.2 Implementation of Finite Difference Algorithm

The wave equation for variable density and tension is:

$$\frac{\partial^2 y(x,t)}{\partial x^2} + \alpha \frac{T_0}{\rho_0} \frac{\partial y(x,t)}{\partial x} = \frac{T_0}{\rho_0} \frac{\partial^2 y(x,t)}{\partial t^2}$$

Now, substituting the finite differences equation derived earlier:

$$\frac{y_{i+1}^n + y_{i-1}^n - 2y_i^n}{\Delta x^2} + \alpha \frac{T_0}{\rho_0} \frac{y_{i+1}^n - y_i^n}{\Delta x} = \frac{T_0}{\rho_0} \frac{y_i^{n+1} + y_i^{n-1} - 2y_i^n}{\Delta t^2}$$

We want to find the value of $y(x,t)$ at the future time step, hence we want to find y_i^{n+1}, so we'll take that on one side of equation and rest to the other. Hence the equation becomes:

$$y_i^{n+1} = \frac{\rho_0}{T_0} \frac{y_{i+1}^n + y_{i-1}^n - 2y_i^n}{\Delta x^2} + \alpha \frac{\Delta t^2}{\Delta x^2} (y_{i+1}^n - y_i^n) - y_i^{j-1} + 2y_i^j$$

By adding the boundary conditions and the initial conditions , and fill the known parameters and iterate whole function using loops and plot the solution.

3.3 Finite Difference Algorithm

Algorithm 1 Finite Difference Algorithm: solving PDE's

Input: $y(x,0), y(0,t), y(L,t), y_x(x,0)$
Output: $PlotWave$ (Solve the PDE using FDM)

function NUMERICALSOL($\mathbf{x}$)
    $NumSol \leftarrow 0$
    $N \leftarrow length(\mathbf{x})$
    for $i \leftarrow 1$ to N **do**
        for $j \leftarrow 1$ to N **do**
            if $x_i < 50$ **then**
                $xi[i,0] = 0.00125 * i$
            end if$x_i >= 50$
            $xi[i,0] = 0.1 - 0.005 * (i - 80)$
            NumSol$[i,2]$ = (NumSol$[i+1,1]$+ NumSol$[i-1,1]$ - 2*NumSol$[i,1]$) + $\alpha * dt * *2/dx*$(NumSol$[i+1,1]$ - NumSol$[i,1]$) - NumSol$[i,0]$
        end for
        return $NumSol$

Chapter 4

Code and Plots

4.1 Code snippet for solving PDE using Finite Differences

Declaring the Initial Condition for the wave being plucked at the centre.

```
def Initialize():                              # Initial conditions
   for i in range(0, 51):
       xi[i, 0] = 0.00125*i
   for i in range (51, 101):
       xi[i, 0] = 0.1 - 0.005*(i - 80)
```

Loop for computing the Finite Differences for the wave equation PDE.

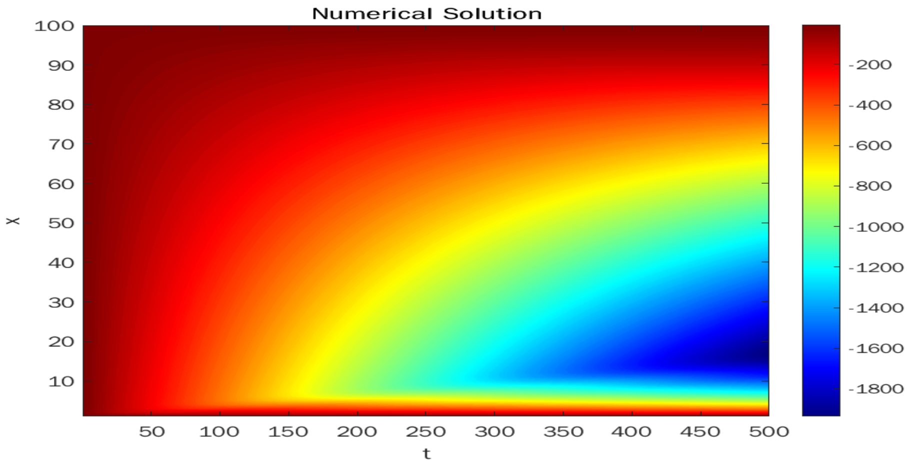
```


def animate(num):
    for i in range(1, 100):
       xi[i,2] = ratio*(xi[i+1,1]+ xi[i-1,1] - 2*xi[i,1]) + alpha*dt**2/dx
   *(xi[i+1,1] - xi[i,1]) - xi[i,0] + 2*xi[i,1]
    line.set_data(k,xi[k,2])           # Data to plot ,x,y
    for m in range (0,101):
       xi[m, 0] = xi[m, 1]             # Recycle array
       xi[m, 1] = xi[m, 2]
    return line
```

- For complete code click here
- Another Implementation click here.
- MATLAB Codes for solving PDE using Finite difference method: here
- Other type of PDEs solved using Finite Difference method: here
- Jupyter Notebook for 1-D Wave equation on a string here
- Jupyter Notebooks for different initial conditions of PDE solved using Finite Difference method here
- Link to the simulation of the wave here

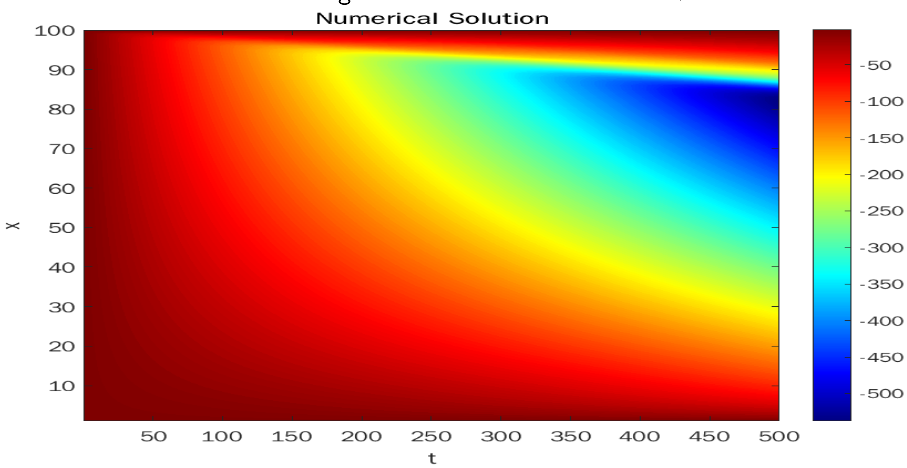
4.2 Plots

The constant $c = \sqrt{\frac{T}{\rho}}$; T represent the tension in the string and ρ linear density of the string. The value of c will increase with lower value of ρ and higher value of T.s. Since c depends on tension T, it means to have a mixture of waves with different amplitudes and frequencies, strings should be stretched to different tensions.



0.5

Figure 4.1: Contour Plot for $x < 0.5$



0.5

Figure 4.2: Contour Plot for $x > 0.5$

Figure 4.3: Contour plot for the Wave Equation

4.2.1 The Length of the String

The frequency of vibration of the string varies greatly with the free length of the string. When the free length is shirt, the vibration is faster and shorter.

4.2.2 The Value of C

The variation in c effectively means variation in the tension and density. Increasing T means an increase in c which is the wave speed. Thus, to have superposition of different waves, it is important to have string stretched to different tension. Related to that, it may be advisable to have strings of different linear densities. This may mean to have string made of different materials or the same material but different cross sectional area.

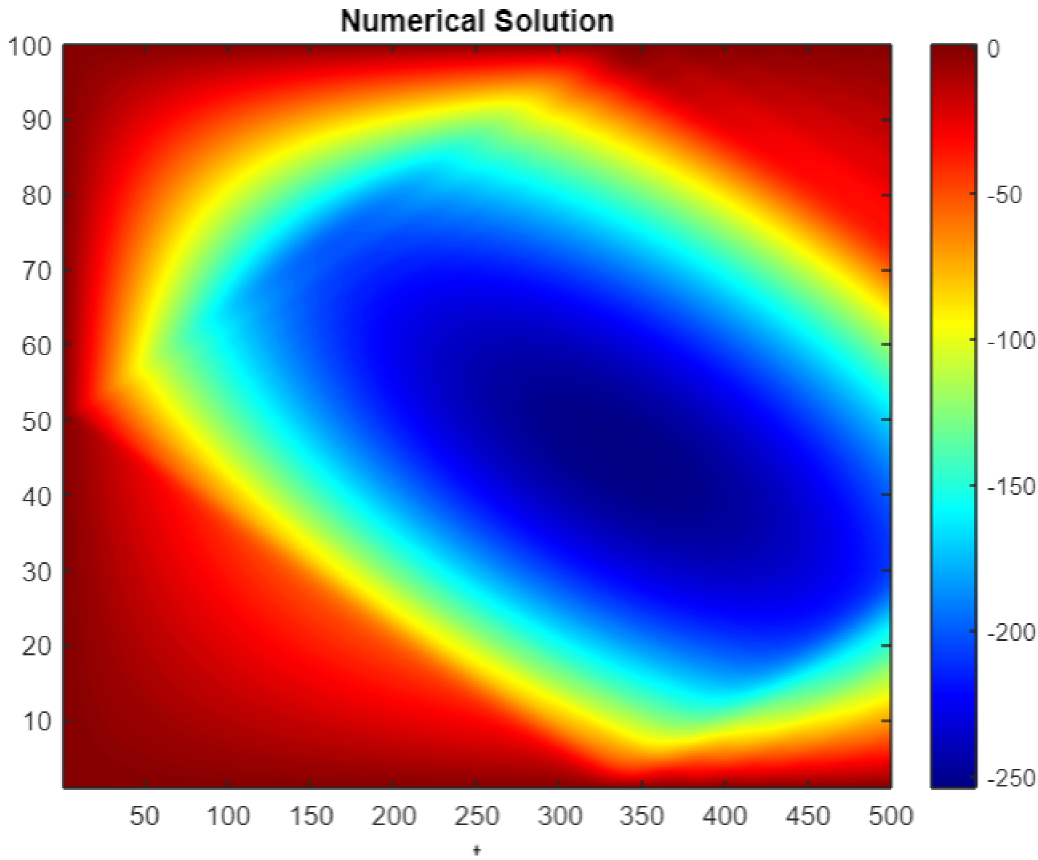


Figure 4.4: A Contour Plot for the Wave equation with the given Initial conditions

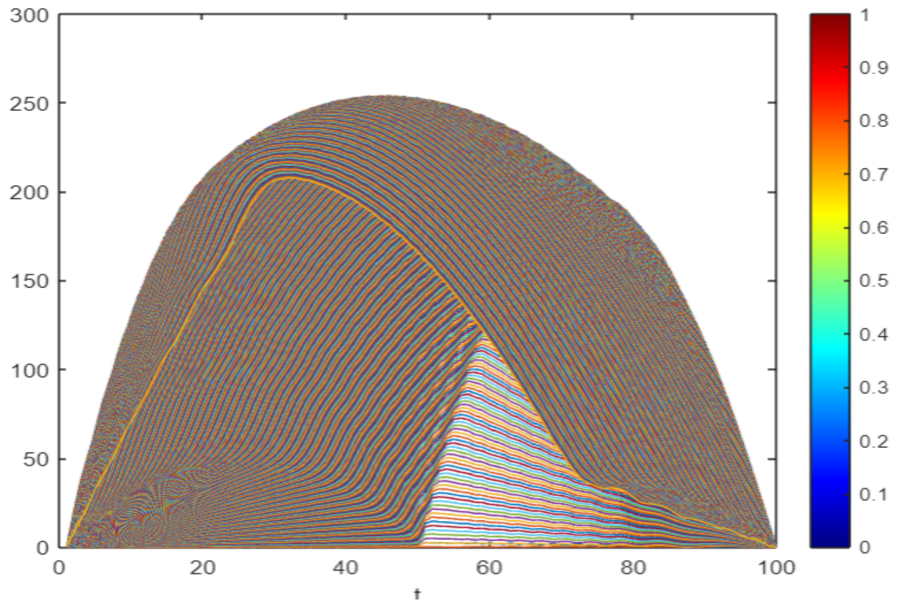


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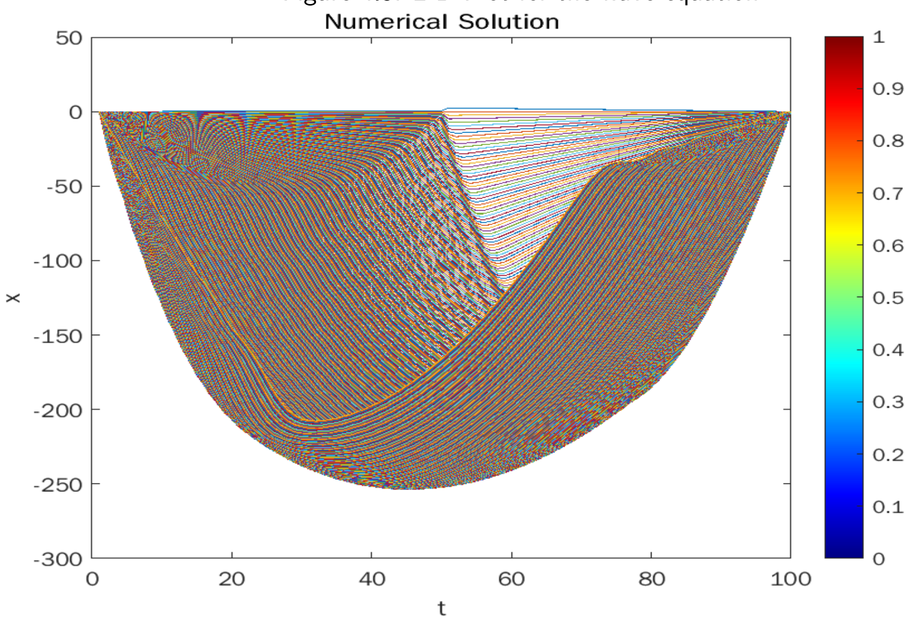
Figure 4.5: 2-D Plot for the wave equation



0.5

Figure 4.6: 2-D Plot for the wave equation for a different Initial Condition

Figure 4.7: 2-D plots for the Wave Equation, a string plucked at the middle.

Chapter 5

Discussion and Analysis

We derive the wave equation in one space dimension that models the transverse vibrations of an elastic string. If such string is placed horizontally between endpoints $x = 0$ and $x = l$, it can freely vibrate within a vertical plane. Generally speaking, it is not true; however, if displacements u(x,t) are small, we can assume that spring motion occurs only within a plane perpendicular to its equilibrium horizontal position.

5.1 Discussion

We try to plot the contour plot for the wave equation for different initial value as seen in Fig. 4.1 and Fig.4.2 , and we see the 2-D plots also mimicking the same behaviour as the contour plots. There's a little asymmetry in the plots due to the initial conditions. Since the Tension and density is not constant throughout the string, we are also given that the displacement for the left half of the string varies differently from that of the right half. Click here to see the simulation

5.2 Review

Perhaps the most straightforward case is observed with the investigation of mechanical vibrations. Suppose that an elastic string of length l is tightly stretched between two supports at the same horizontal level, which we identify with the x-axis. Then its endpoints may be taken as $x = 0$ and $x = l$. The elastic string maybe a guitar or violin string, a guy wire, or possibly an electric power line. The positions of points on the string can be described by the displacement, which we denote by $u(x, t)$, from the equilibrium horizontal position. If damping effects, such as air resistance, are neglected, and if the magnitude of the motion is not too large, then the displacement function satisfies the partial differential equation (called one-dimensional wave equation)

$$u_{tt} = c^2 u_{xx}$$

For a more general case, with no variable density and Tension, in the domain $0 < x < l$ and $0 < t < \infty$. The constant coefficient c^2 is given by:

$$c^2 = \frac{T}{\rho}$$

where T is the tension (force) in the string, and ρ is the mass per unit length of the string material (density).

Chapter 6

References

- PROFESSOR D. M. CAUSON PROFESSOR C. G. MINGHAM INTRODUCTORY FINITE DIFFERENCE METHODS FOR PDES
- Christian Grossmann; Hans-G. Roos; Martin Stynes (2007). Numerical Treatment of Partial Differential Equations. Springer Science Business Media. p. 23. ISBN 978-3-540-71584-9.
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- ON FINITE DIFFERENCES ON A STRING PROBLEM Mango, J.M., C. Eryenyu and S.E. Rugeihyamu
- Link to all codes here

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Appendix A

Definition and Properties of Order

A.1 Definition of $O(h^n)$

For our purpose $f(h) = O(h^n)$ means

$$lim_{h\to 0}\frac{f(h)}{h^n} = C$$

where C is a non-zero constant. Eg. $500h^6 + 3h^4 - 2h = O(h)$ because

$$lim_{h\to 0}\frac{500h^6 + 3h^4 - 2h}{h} = 2$$

A.2 The Meaning of $O(h^n)$

If $f(h) = O(h^n)$then, for small h, we get

$$\frac{f(h)}{h^n} = C$$

, which gives

$$f(h) = Ch^n$$

Above equation says that for small h, an error which is $O(h^n)$ is proportional to h^n. In particular if the error is $O(h)$ then it is proportional to h which means that halving h halves the error. If the error is $O(h^2)$ then it is proportional to h^2 which means that halving h reduces the error by a factor of $2^2 = 4$.