

Lecture 2: Quantum Computation Review

Nature behaves in non-local manner.

I think that a particle must have a separate reality independent of measurements. That is, an electron has spin, location and so forth even when it is not being measured. I like to think the moon is there even if I am not looking at it

— Albert Einstein

... experiments have now shown that what bothered Einstein is not a debatable point but the observed behaviour of the real world

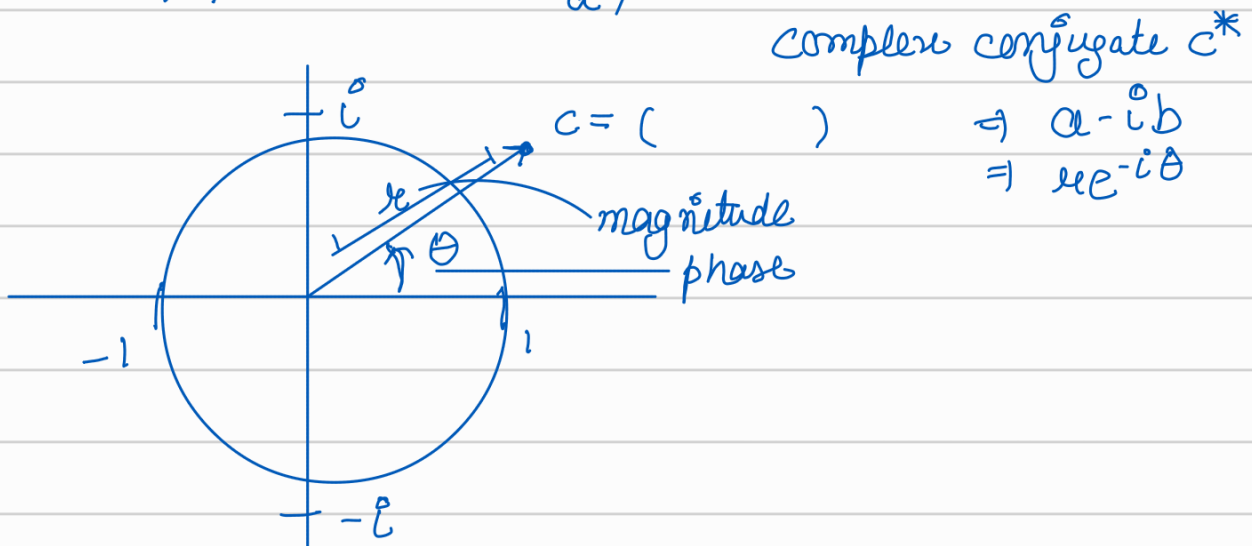
— David Mermin

1) Linear Algebra

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_d \end{pmatrix} \in \mathbb{C}^d$$

ket $|\psi\rangle$

$$\begin{aligned} a + ib &\in \mathbb{C} \\ re^{i\theta} &\in \mathbb{C} \\ a, b, r, \theta &\in \mathbb{R} \end{aligned}$$



$$c = r(\cos\theta + i\sin\theta) \quad \text{Euler's formula}$$

$$\rightarrow |c| = \sqrt{c^*c}$$

$$\text{or } |re^{i\theta}| = |r| |e^{i\theta}|$$

$$\Rightarrow |r| |e^{-i\theta} e^{i\theta}|$$

$$\Rightarrow |r|$$

$$\begin{aligned} c + c^* &= 2\text{Re}(c) \\ c - c^* &= 2i\text{Im}(c) \\ cc^* &= |c|^2 \end{aligned}$$

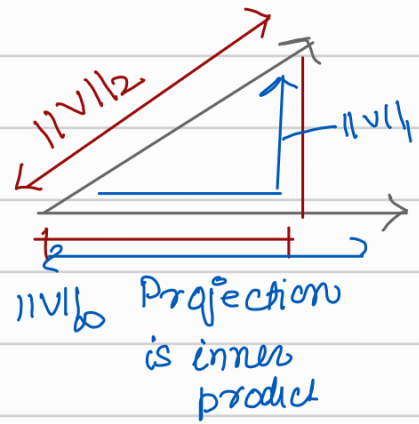
"bra ψ "

$$\langle \psi | = (\psi_1^* \ \psi_2^* \ \dots \ \psi_d^*)$$

Inner product $\langle \psi | \cdot | \psi \rangle = \langle \psi | \psi \rangle \in \mathbb{C}$

$1 \times d \quad d \times 1$

$$\Rightarrow \sum_{i=1}^d \psi_i^* \psi_i$$



Euclidean Norm $\| |\psi\rangle \|_2 = \sqrt{\langle \psi | \psi \rangle}$

length of the vector

$$\| |\psi\rangle \|_1 = \sum_i |\psi_i| \quad \text{one norm}$$

$$\| |\psi\rangle \|_\infty = \max_i |\psi_i| \quad \text{infinity norm}$$

Orthonormal Basis set $\{ |\psi_i\rangle \} \subseteq \mathbb{C}^d$ s.t. $\langle \psi_i | \psi_j \rangle = \delta_{ij}$

δ Kronecker delta

standard basis / computational basis

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \quad \dots \quad |d-1\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

$\begin{cases} 1 & \text{if } i=j \\ 0 & \text{o/w} \end{cases}$

Any $|\psi\rangle \in \mathbb{C}^d$ as $|\psi\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle \in \mathbb{C}^d$

if $\| |\psi\rangle \|_2 = 1 \Rightarrow \sum_i |\alpha_i|^2 = 1$

normalized

Linear Maps and Matrices map $\Phi: \mathbb{C}^d \rightarrow \mathbb{C}^d$ is linear if

$$\Phi \left(\sum_i \alpha_i |\psi_i\rangle \right) = \sum_i \alpha_i \Phi(|\psi_i\rangle) \quad (\text{linearity})$$

$\mathcal{L}(x, y) \rightarrow$ set of linear maps from $x \mapsto y$

$$\mathcal{L}(x) \rightarrow \hat{\Phi} : x \mapsto x$$

matrix representation of $\Phi : \mathbb{C}^d \mapsto \mathbb{C}^d$, A_Φ is a $d \times d$ matrix

$$A = \begin{bmatrix} | & | & & | \\ \Phi(|0\rangle) & \Phi(|1\rangle) & \dots & \Phi(|d-1\rangle) \\ | & | & & | \end{bmatrix}_{d \times d} \in \mathbb{C}^{d \times d}$$

EXERCISE $\Phi(|0\rangle) = |1\rangle$ $\Phi(|1\rangle) = |0\rangle$

$$A_\Phi = \begin{pmatrix} |1\rangle & |0\rangle \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \underline{\underline{\text{x gate}}}$$

$$A|i\rangle = i^{\text{th}} \text{ column of } A.$$

$$x|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow |1\rangle$$

Matrix multiplication

$$AB_{ij} = \sum_{k=1}^d A(i, k) B(k, j)$$

Recall! matrix multiplication $AB \neq BA$ in general
(this keeps matter stable)

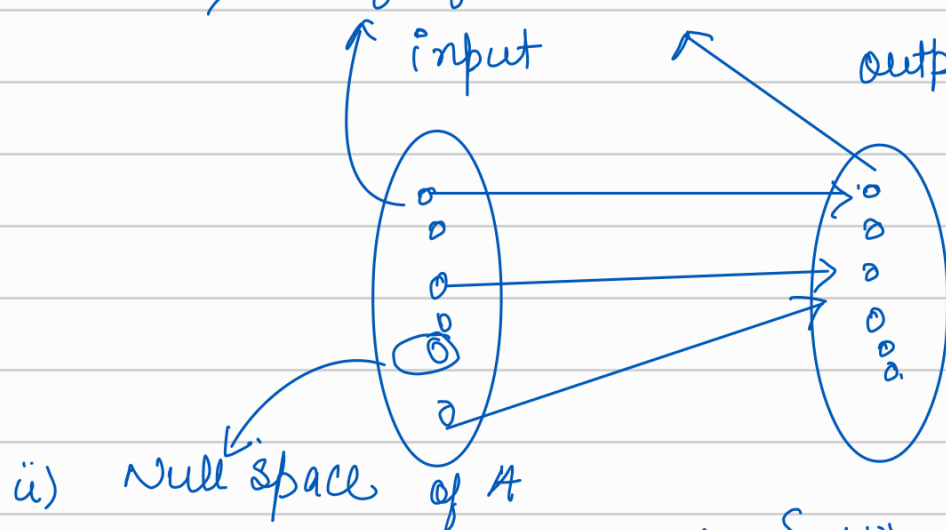
if $AB = BA$, say A and B commute.

$$[A, B] = AB - BA = 0$$

EXERCISE

$XZ \neq ZX$ ($XZ = -ZX$) They anti commute.

Spaces i) Image of A : $\text{Im}(A) = \{ |\psi\rangle \in \mathbb{C}^d \mid \exists |\phi\rangle \in \mathbb{C}^d \text{ s.t. } A|\phi\rangle = |\psi\rangle \}$



$$\text{Null}(A) = \{ |\psi\rangle \in \mathbb{C}^d \mid A|\psi\rangle = 0 \}$$

$$\text{rank}(A) = \dim(\text{Im}(A))$$

Rank-nullity theorem

$$\text{rank}(A) + \dim(\text{null}(A)) = d \text{ (full dimension)}$$

Matrix operations a) $A^\dagger = (A^T)^*$ conjugate + transpose, adjoint, dagger.

recall $(AB)^\dagger = B^\dagger A^\dagger$ (flips the order)

b) Trace $(A) = \text{Sum of all diagonal terms}$ $\sum_{i=1}^d A_{i,i}$
Trace is linear

$$\text{Tr}: \mathcal{L}(\mathbb{C}^d) \mapsto \mathbb{C}$$

take a matrix

gives a single number.

and is cyclic

$$\text{Tr}(ABC) = \text{Tr}(BCA)$$

the implication is $\text{Tr}(AB) = \text{Tr}(BA)$ always

Outer Product

$|\psi\rangle, |\phi\rangle \in \mathbb{C}^d \rightarrow$ outer product
 $|\psi\rangle\langle\phi| \in L(\mathbb{C}^d)$
 $d \times 1 \quad 1 \times d$

E.g. $|1\rangle\langle 0| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$|i\rangle\langle j| \rightarrow 1$ in position (i,j) , 0 otherwise

Trick $\langle i|A|j\rangle = \langle i| \left(\sum_{i',j'} A(i',j') |i'\rangle\langle j'| \right) |j\rangle$
 $1 \times d \quad d \times d \quad d \times 1$

std. basis are orthonormal

$$\Rightarrow \sum_{i',j'} A(i',j') \underbrace{\langle i|i'\rangle}_{\delta_{ii'}} \underbrace{\langle j'|j\rangle}_{\delta_{jj'}}$$

$$\Rightarrow A(i,j)$$

(outer product take vectors together and make matrices out of it)

EXERCISE 9: $\text{Tr}(A) = \sum_i A(i,i)$

$$\sum_i \langle i|A|i\rangle$$

Extracting out the diagonal entries

Eigenvalues and Eigenvectors

A matrix $A \in L(\mathbb{C}^d)$ have eigenvector $|\psi\rangle$ if

$$A|\psi\rangle = \lambda|\psi\rangle$$

$\nearrow$ eigenvalues

An operator A is normal iff $AA^\dagger = A^\dagger A$ iff A has a spectral decomposition

$$A = \sum_i \lambda_i |i\rangle\langle i|$$

$\uparrow$ eigenvalues

$\nwarrow$ eigen vectors

if A is normal then the eigenvectors can be chosen as

if A is normal then the eigenvectors can be chosen as orthonormal basis.

if A is normal then, (because spectral decomposition)

$$\text{Tr}(A) = \text{Tr}\left(\sum_i \lambda_i \langle \lambda_i | A | \lambda_i \rangle\right)$$

(trace is linear)

$$\Rightarrow \sum_i \lambda_i \text{Tr}(|\lambda_i\rangle\langle\lambda_i| A |\lambda_i\rangle\langle\lambda_i|)$$

$$\Rightarrow \sum_i \lambda_i \langle \lambda_i | A | \lambda_i \rangle \quad (\text{trace is cyclic})$$

Trace of an operator is always the sum of its eigenvalues.

EXERCISE: If A is diagonalizable (normal) then prove that

$$\text{rank}(A) = \text{number of non-zero eigenvalues of } A$$

Rank of a diagonalizable matrix is equal to the number of non-zero eigenvalues

If A is diagonalizable then $\exists P$ invertible matrix such that

$$P^{-1}AP = D$$

where D is diagonal matrix having eigenvalues of A on its diagonal and each eigenvalue appears as many times as its multiplicity.

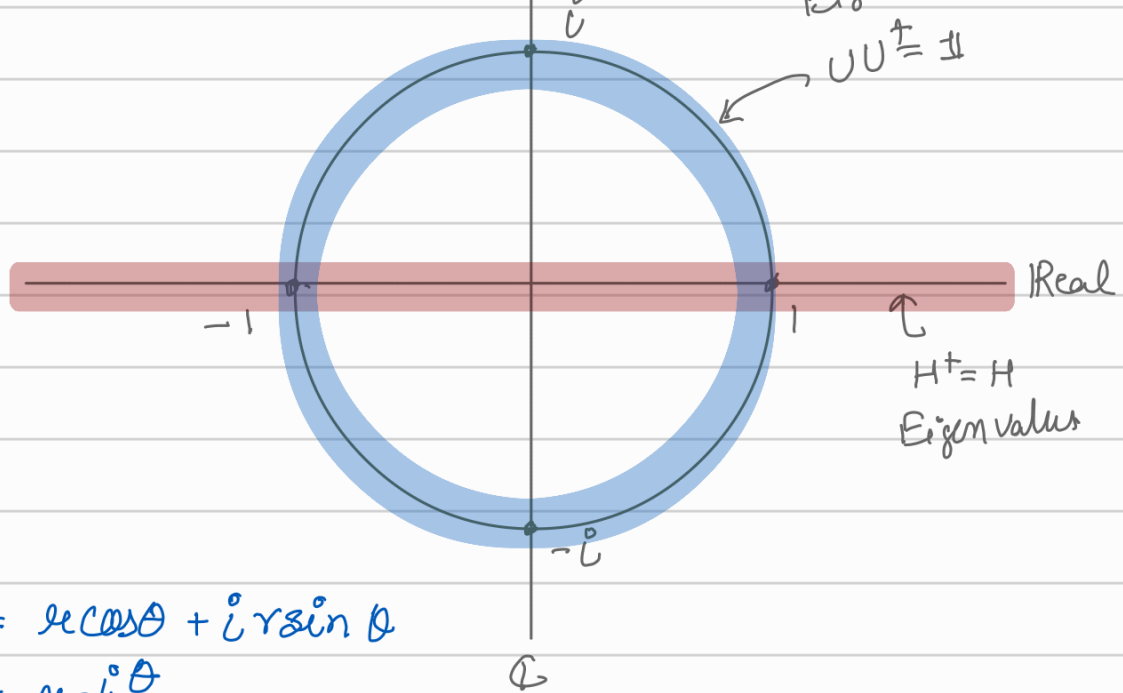
$$\kappa(A) = \kappa(P^{-1}AP) = \kappa(D)$$

Important classes of Matrices

1) unitary matrices : $U \in \mathcal{L}(\mathbb{C}^d)$ if $UU^\dagger = \mathbb{I} = U^\dagger U$

EXERCISE: Eigenvalues of unitary operator are always on the unit circle.

Eigenvalues



$$x + iy = r \cos \theta + i r \sin \theta$$

$$= r e^{i\theta}$$

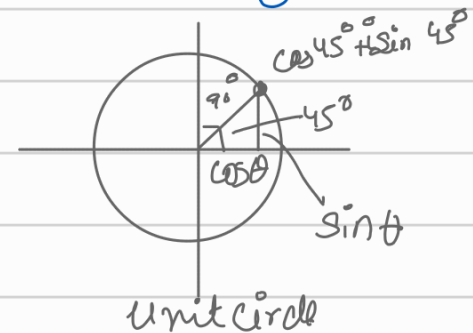
$$r = |x + iy| = \sqrt{(x + iy)(x - iy)} \Rightarrow \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Let's see how can we write complex numbers using these notations.

$$\cos 45^\circ + i \sin 45^\circ \Rightarrow \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}$$

$$\Rightarrow e^{i\frac{\pi}{4}}$$



Unitary matrices preserve norm of a vector
eigenvalue λ eigenvector $|\psi\rangle$

$$U|\psi\rangle = \lambda|\psi\rangle \sim \text{Eigenvalue equation}$$

unitary matrices preserve norm

$$\| |\psi\rangle \| = \| U|\psi\rangle \|$$

$$\| |\psi\rangle \| = \| \lambda|\psi\rangle \|$$

$$\sqrt{\langle \psi | \psi \rangle} = \sqrt{\langle \lambda \psi | \lambda \psi \rangle}$$

$$= \sqrt{\langle \psi | \lambda^* \lambda | \psi \rangle}$$

$$= \sqrt{\lambda^* \lambda \langle \psi | \psi \rangle}$$

$$\sqrt{\langle \psi | \psi \rangle} \Rightarrow |\lambda| \sqrt{\langle \psi | \psi \rangle}$$

absolute value of complex number, how far is it from the center of the circle

$$|z|=1$$

~ hence the eigenvalues of a unitary operator lie on the complex number

Unitary matrices are high dimensional generalization of unit complex numbers.

Set of unitary : $\mathcal{U}(X)$

ii) Hermitian $H \in \mathcal{L}(\mathbb{C}^d)$ if $H = H^\dagger$

set of Hermitian operator $\text{Herm}(X)$

their eigen values are always $\mathbb{R}$

and they generalize real number.

iii) Positive (semi-) definite : Hermitian + ^{semi-def} non-negative or ^{strictly positive} strictly positive.
 $\text{Pos}(X)$

If one of your eigenvalue is 0, you're not invertible. $H \geq 0$ $H > 0$
semi definite definite

generalize set of non-negative and positive reals.

iv) Orthogonal Projections : $\Pi \in \mathcal{L}(\mathbb{C}^d)$ s.t
 Π is Hermitian and $\Pi^2 = \Pi$

$\Pi^2 = \Pi \Rightarrow$ eigenvalues $\in \{0, 1\}$

spectral decomp: $\Pi = \sum_i |\psi_i\rangle\langle\psi_i|$ for $\{|\psi_i\rangle\}$ orthonormal basis.

rank-1 proj : $|\psi\rangle\langle\psi|$

only one - non neg

eigenvalue

Some Revision on Maps

Maps $\underline{\Phi}: \mathbb{C}^d \mapsto \mathbb{C}^d$ are linear, meaning

$$\sum_i \alpha_i |\psi_i\rangle \in \mathbb{C}^d \text{ that}$$

$$\underline{\Phi}(\sum_i \alpha_i |\psi_i\rangle) = \sum_i \alpha_i \underline{\Phi}|\psi_i\rangle$$

hence linear maps.

$\mathcal{L}(\pi, \gamma)$ they have matrix representation

$$A_{\underline{\Phi}} = [\underline{\Phi}|0\rangle, \underline{\Phi}|1\rangle, \dots, \underline{\Phi}|d-1\rangle]$$

we define the i^{th} column of $A_{\underline{\Phi}}$ as $\underline{\Phi}|i\rangle$ too $|i\rangle$'s the standard basis for $\mathbb{C}^d$.

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The image of a matrix A is the set of all possible output vectors under the action of A .

$$\text{Im}(A) := \{|\psi\rangle \in \mathbb{C}^d \mid |\psi\rangle = A|\phi\rangle \text{ for some } \phi \in \mathbb{C}^d\}$$

The rank of A is the dimension of its image. i.e $\dim(\text{Im}(A))$

$$\text{Null}(A) := \{|\psi\rangle \in \mathbb{C}^d \mid A|\psi\rangle = 0\}$$

Rank-Nullity Theorem

$$\dim(\text{Null}(A)) + \dim(\text{Im}(A)) = d$$

✱

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A trace is a linear map $\text{Tr}: \mathcal{L}(\mathbb{C}^d) \mapsto \mathbb{C}$ summing the entries on the diagonal of A .

$$\text{Tr}(A) = \sum_{i=1}^d A(i, i)$$

Trace is cyclic

$$\text{Tr}(ABC) = \text{Tr}(CAB)$$

Operator Functions

How to apply $f: \mathbb{R} \rightarrow \mathbb{R}$ to matrices. $f(x) = \sqrt{x}$
 for H being a Hermitian operator $H \in \text{Herm}(\mathbb{C}^d)$ with
 spectral decomp $H = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$

$$f: \text{Herm}(\mathbb{C}^d) \rightarrow \text{Herm}(\mathbb{C}^d)$$

$\rightarrow$ apply to scalar.

$$\Rightarrow f(H) = \sum_i f(\lambda_i) |\lambda_i\rangle\langle\lambda_i|$$

EXERCISE $f(x) = e^x$. Prove $f(H) = e^H = I + H + \frac{H^2}{2!} + \frac{H^3}{3!} + \dots$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

We know $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Since H is a Hermitian matrix, we can diagonalize as
 $H = U \Lambda U^\dagger$ where U is a unitary matrix and
 Λ is a diagonal matrix containing the eigenvalues of H .

$$f(H) = f(U \Lambda U^\dagger) = U f(\Lambda) U^\dagger$$

Now for $f(\Lambda) = e^\Lambda = \sum_{n=0}^{\infty} \frac{\Lambda^n}{n!}$

Since Λ is a diagonal matrix, Λ^n is also a diagonal matrix
 with the i -th diagonal element raised to power of n , denoted
 as λ_i^n , where λ_i is the i -th eigenvalue of H . $\therefore$

$$e^\Lambda = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} e^{\lambda_1} & 0 & \dots & 0 \\ 0 & e^{\lambda_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\lambda_n} \end{pmatrix}$$

Now, substitute e^λ back into the expression

$$f(H) = U f(\Lambda) U^\dagger = U \left(\sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} e^{\lambda_1} & 0 & \dots & 0 \\ 0 & e^{\lambda_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\lambda_n} \end{pmatrix} \right) U^\dagger$$

By regrouping

$$f(H) = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} U e^{\lambda_1} U^\dagger & 0 & \dots & 0 \\ 0 & U e^{\lambda_2} U^\dagger & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & U e^{\lambda_n} U^\dagger \end{pmatrix} = e^H$$

EXERCISE $f(x) = \sqrt{x}$, $\Rightarrow \sqrt{I} = \sqrt{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \Rightarrow \begin{bmatrix} \sqrt{1} & 0 \\ 0 & \sqrt{1} \end{bmatrix}$

$\sqrt{I} \leftarrow \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

Basic Quantum Computation

Pure State computation - d -dimensional pure state $|\psi\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle$
 $\alpha_i \in \mathbb{C}^d$

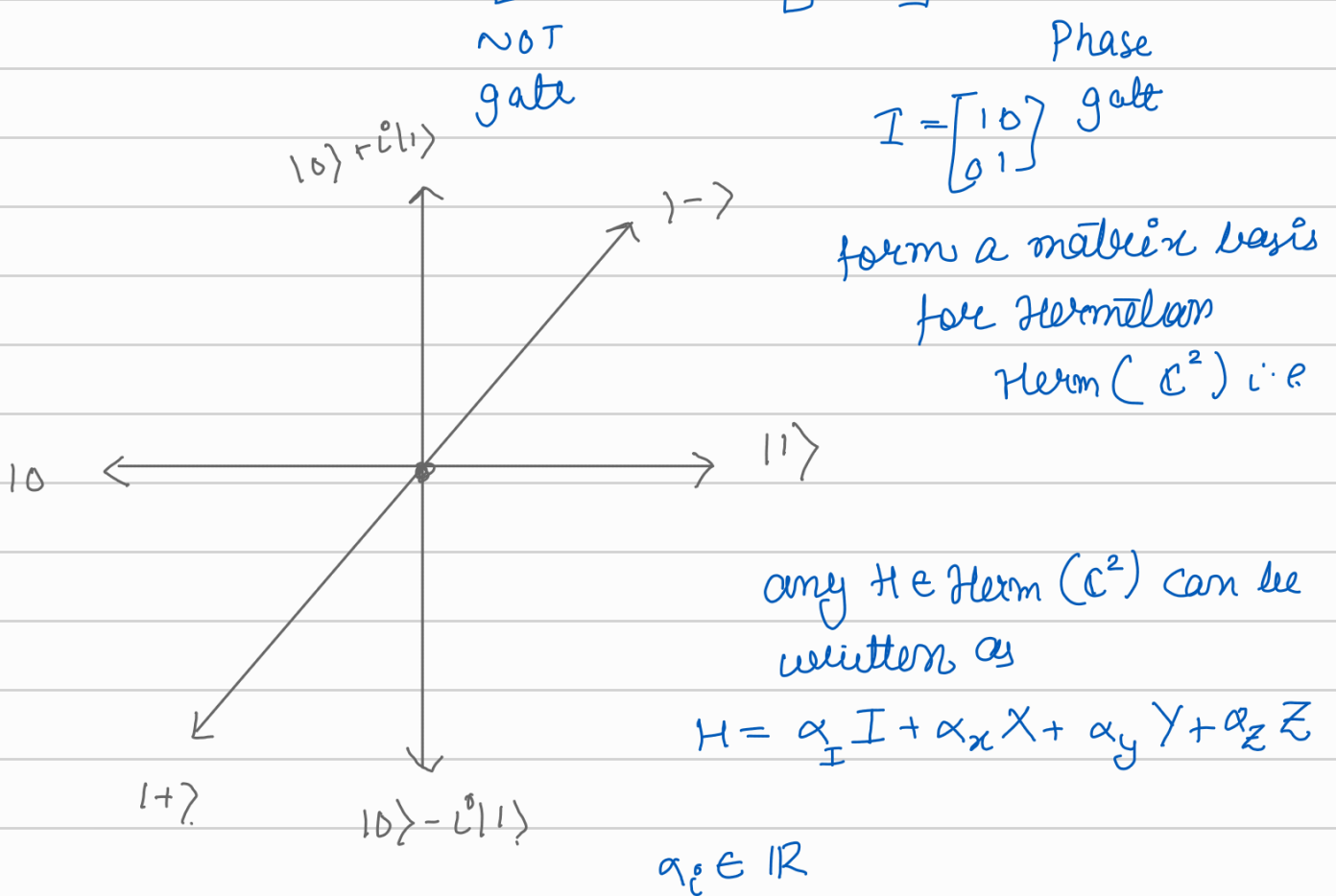
$$\|\psi\|_2 = 1$$

d -dim = qudit

2-dim = qubit

Quantum gates - set of allowable operators are $U \in \mathcal{U}$, such that $UU^\dagger = I$. The gates are all reversible.

for $d=3$, we see $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$



Hadamard

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

introduces superposition

$$H|0\rangle = |+\rangle \Rightarrow \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$$

$$H|1\rangle = |-\rangle \Rightarrow \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$$

Relative Phase

$$Z|0\rangle = |0\rangle$$

$$Z|1\rangle = -|1\rangle$$

phase

$$Z|+\rangle = \frac{1}{\sqrt{2}} Z|0\rangle + \frac{1}{\sqrt{2}} Z|1\rangle = |-\rangle$$

$$Z|-\rangle = |+\rangle$$

NOT gate

Global phase

$$e^{i\theta}|0\rangle + e^{i\theta}|1\rangle \Rightarrow e^{i\theta} (|0\rangle + |1\rangle)$$

we cannot detect global phase in measurement.

Quantum circuit model



$$X|\psi\rangle$$



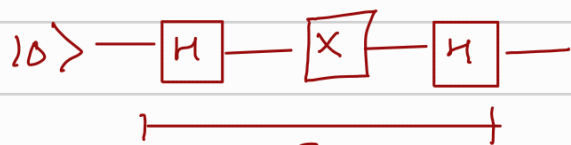
$$H|\psi\rangle$$



$$HX|\psi\rangle$$

EXERCISE What gate does this circuit simulate?

Hint: spectral decompose X, Z



Soln:

$$X = |0\rangle\langle 1| + |1\rangle\langle 0|$$

$$H = \frac{1}{\sqrt{2}} (|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$$

$$HXH|\psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|) (|0\rangle\langle 1| + |1\rangle\langle 0|) \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$$

we get

$$HXH|\psi\rangle = \frac{1}{\sqrt{2}} (|1\rangle - |0\rangle) = -Z|\psi\rangle$$

Hence, the circuit HXH simulates a Z gate.

Composite Quantum Systems

we make use tensor product, $\otimes : \mathbb{C}^{d_1} \times \mathbb{C}^{d_2} \rightarrow \mathbb{C}^{d_1 \times d_2}$

formally if we have.

$$\mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2 \cong \mathbb{C}^{2^n}$$

$$(|\psi\rangle \otimes |\varphi\rangle)(i,j) = \psi_i \varphi_j \quad \underbrace{\quad}_{n \text{ copies}}$$

Eg.
$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a \begin{pmatrix} c \\ d \end{pmatrix} \\ b \begin{pmatrix} c \\ d \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}$$

Properties of tensor product

$$(|a\rangle + |b\rangle) \otimes |c\rangle = |a\rangle \otimes |c\rangle + |b\rangle \otimes |c\rangle$$

$$|a\rangle \otimes (|c\rangle + |d\rangle) = |a\rangle \otimes |c\rangle + |a\rangle \otimes |d\rangle$$

$$c(|a\rangle \otimes |c\rangle) = (c|a\rangle) \otimes |c\rangle = |a\rangle \otimes (c|c\rangle)$$

$$(|a\rangle \otimes |c\rangle)^+ = |a\rangle^+ \otimes |c\rangle^+ \Rightarrow \langle a| \otimes \langle c|$$

$$(\langle a| \otimes \langle c|)(|b\rangle \otimes |d\rangle) = \langle a|b\rangle \langle c|d\rangle$$

Quantum Entanglement:

$$|\psi\rangle, |\phi\rangle \rightarrow \mathbb{C}^d \mapsto |\psi\rangle \otimes |\phi\rangle \rightarrow \mathbb{C}^{d^2}$$

but the converse is not generally true

$$\exists |\eta\rangle \in \mathbb{C}^{d^2} \text{ s.t. } \nexists |\psi\rangle, |\phi\rangle \rightarrow \mathbb{C}^d \text{ s.t. } |\psi\rangle \otimes |\phi\rangle = |\eta\rangle$$

↳ entangled states.

Schmidt decompositions $\forall |\eta\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, we can always

$$|\eta\rangle = \sum_{i=0}^{\min(d_1, d_2)-1} s_i |a_i\rangle |b_i\rangle \leftarrow \text{Schmidt vectors (orthonormal)}$$

↳ Schmidt coefficient, $s_i \geq 0$

Schmidt rank: # of non-zero Schmidt coefficients.

$|\eta\rangle$ is entangled iff Schmidt rank > 1 .

E.g: Bell Basis in $\mathbb{C}^2 \otimes \mathbb{C}^2$

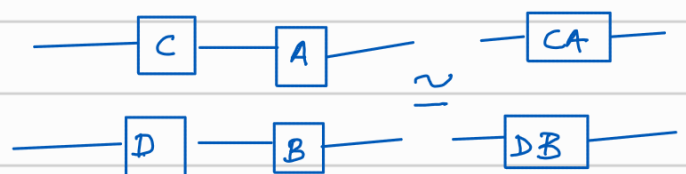
$$|\phi^+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \quad |\phi^-\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle)$$

$$|\psi^+\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) \quad |\psi^-\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle)$$

Two Qubit Gates $\begin{cases} \text{tensor product of 1-qubit gate} \\ \text{genuine 2-qubit gate} \end{cases} \quad X \otimes Z \in U(\mathbb{C}^4)$

i) Tensor product of 1-qubit gate

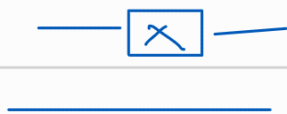
$$\begin{array}{ccccccc} (A \otimes B)(C \otimes D) & = & AC & \otimes & BD \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ q_1 & q_2 & q_1 & q_2 & q_1 & q_2 & \end{array}$$



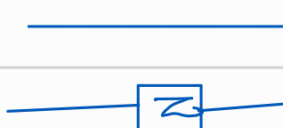
$$\text{Tr}((x \otimes x)(x \otimes x)) = \text{Tr}(x^2 \otimes x^2) \Rightarrow \text{Tr}(I \otimes I) \Rightarrow \text{Tr}(I) \text{Tr}(I) \\ \Rightarrow 2 \times 2 = 4$$

$$\text{Tr}(A \otimes B) = \text{Tr}(A) \cdot \text{Tr}(B)$$

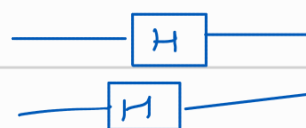
Circuits. $x \otimes I$



$I \otimes Z$



$H \otimes H$



ii) Genuine 2-qubit gate $U \in \text{UC}(\mathbb{C}^4)$
Controlled - NOT (CNOT)

$$\begin{aligned} \text{CNOT}|00\rangle &= |00\rangle \\ \text{CNOT}|01\rangle &= |01\rangle \\ \text{CNOT}|10\rangle &= |11\rangle \\ \text{CNOT}|11\rangle &= |10\rangle \end{aligned}$$

$$\text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\text{CNOT} = \begin{bmatrix} [I] & 0 \\ 0 & [X] \end{bmatrix}$$

→ Measurement.

Measurement is done to extract information from the system.

→ basic type: projective / von-Neumann measurement

they are just set of projectors $B = \{ \pi_i \}_{i=0}^m$

$$\sum_{i=0}^m \pi_i = I \quad \text{completeness relation}$$

If each $\pi_i = |\psi_i\rangle\langle\psi_i|$ i.e. $\text{rank}(\pi_i) = 1$

say measure in the basis $\{|\psi_i\rangle\}$

Given state $|\psi\rangle \in \mathbb{C}^d$ and measurement $B = \{ \pi_i \}_{i=0}^m \subseteq \mathbb{C}^d$

$$i) P(\text{outcome } i | |\psi\rangle) = \text{Tr}(\pi_i |\psi\rangle\langle\psi| \pi_i) \\ = \text{Tr}(\pi_i^2 |\psi\rangle\langle\psi|) \Rightarrow \text{Tr}(\pi_i |\psi\rangle\langle\psi|)$$

ii) Post measurement state

$$|\psi'\rangle = \frac{\pi_i |\psi\rangle}{\|\pi_i |\psi\rangle\|_2} \Rightarrow \frac{\pi_i |\psi\rangle}{\sqrt{\langle\psi| \pi_i |\psi\rangle}}$$

mixed state Quantum Computation

A mixed state represented by a density operator $\rho \in \mathcal{L}(\mathbb{C}^d)$

s.t. (i) $\rho_i \geq 0$ (PSD)

(ii) $\text{Tr}(\rho) = 1$

quantum uncertainty - entanglement

classical uncertainty - randomness

Spectral decomposition $\rho = \sum_i \rho_i |\psi_i\rangle\langle\psi_i|$

$$\sum_i \rho_i = 1 \text{ by (ii)}$$

Maximally mixed state $\rho \in \mathcal{L}(\mathbb{C}^d)$ s.t. $\rho = \frac{\mathbb{I}}{d}$ - state of $\overline{d}$ no information

$\forall$ orthonormal basis $\{|\psi_i\rangle\}_{i=1}^d$, $\sum_{i=1}^d |\psi_i\rangle\langle\psi_i| = \mathbb{I}$

Partial Trace Operation

$$\text{Tr}_B : \mathcal{L}(\mathbb{C}^d_A) \times \mathcal{L}(\mathbb{C}^d_B) \longrightarrow \mathcal{L}(\mathbb{C}^d_A)$$

+ tracing out the second system.

$$\text{Tr}_B(\rho_{AB}) = \sum_{i=1}^d \left(\mathbb{I}_A \otimes \langle i|_B \right) \rho_{AB} \left(\mathbb{I}_A \otimes |i\rangle_B \right)$$

EXERCISE : Prove that $\text{Tr}_B(\rho_B \otimes \rho_A) = \rho_A \cdot \text{Tr}(\rho_B) = \rho_A$

Given ρ_A and ρ_B are density matrices

we know $\text{Tr}(\rho_A) = 1$ $\text{Tr}(\rho_B) = 1$

$$\text{Tr}_B(\rho_A \otimes \rho_B) = \sum_i (\rho_A \otimes \rho_B)_{ii}$$

$$\Rightarrow \sum_i \sum_j \rho_A_{ij} \rho_B_{ij}$$

$$\Rightarrow \sum_i \rho_A_{ii}$$

Separable states Pure states $|\psi\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ are not entangled,
and are separable iff $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$
 $\begin{matrix} \in & \in \\ \mathbb{C}^{d_1} & \mathbb{C}^{d_2} \end{matrix}$

Extending this idea for mixed states.

A bipartite density matrix $\rho \in \mathcal{L}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ is separable if

$$\rho = \sum_i \beta_i |\psi_i\rangle\langle\psi_i| \otimes |\varphi_i\rangle\langle\varphi_i|$$