Dirac notations in state space

Euclidean space ~ 3D ~ Real

State space ~ ND ~ Complex

the vector space in which quantum systems live. just like Euclidean space is for classical systems.

both the spaces Inner product.

A vector space equipped with an inner product is an Hilbert space.

Postulate 1: The state of a physical system is characterized by a state vector that belongs to a complex vector space V, called the state space of the system

Euclidean 3D space $\mathbb{R}^3$ | State space ~ Dirac Not.

$\underline{r}$ "vector" | $|\psi\rangle$ "ket"

Scalar product in Euclidean space

$$SP(\underline{r}_1, \underline{r}_2) = \underline{r}_1 \cdot \underline{r}_2 = c \in \mathbb{R} \qquad (x_1 \ y_1 \ z_1)\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = c$$

① conjugation $\quad SP(\underline{r}_1, \underline{r}_2) = SP(\underline{r}_2, \underline{r}_1)$

$$\underline{r}_1 \cdot \underline{r}_2 = \underline{r}_2 \cdot \underline{r}_1$$

$c = x_1 y_1 + x_2 y_2 + x_3 y_3$

(addition is not defined, so this means they must be in diff V space)

② linearity $\quad SP(\underline{r}_1, a\underline{r}_2) = a\,SP(\underline{r}_1, \underline{r}_2)$

$$\underline{r}_1 \cdot (a\underline{r}_2) = a(\underline{r}_1 \cdot \underline{r}_2)$$

$$SP(\underline{r}_1, \underline{r}_2 + \underline{r}_3) = SP(\underline{r}_1, \underline{r}_2) + SP(\underline{r}_1, \underline{r}_3)$$

③ positive $\quad SP(\underline{r}_1, \underline{r}_1) \geq 0 \Rightarrow \underline{r}_1 \cdot \underline{r}_1 \geq 0 \Rightarrow |\underline{r}_1|^2 \geq 0$

$$\underline{r}_1 \cdot \underline{r}_1 = 0 \text{ iff } \underline{r}_1 = 0$$

Scalar product in state space

$SP(|\psi\rangle, |\varphi\rangle) = c \quad , \quad c \in \mathbb{C}$

(1) conjugation $\quad SP(|\psi\rangle, |\varphi\rangle) = [SP(|\varphi\rangle\, |\psi\rangle)]^*$

(2) Linear but only in the second argument

$$SP(|\psi\rangle, \underline{a|\varphi\rangle}) = a\,SP(|\psi\rangle, |\varphi\rangle)$$

$$SP(|\psi\rangle, \underline{\underline{|\varphi\rangle + |\gamma\rangle}}) = SP(|\psi\rangle, |\varphi\rangle) + SP(|\psi\rangle, |\gamma\rangle)$$

only in second argument

Anti linear in the first argument

$$SP(a|\psi\rangle, |\varphi\rangle) = [SP(|\varphi\rangle, a|\psi\rangle)]^*$$

move it to second argument

that adds the conjugation

$$\Rightarrow a^* [SP(|\varphi\rangle, |\psi\rangle)]^* \Rightarrow a^* SP(|\psi\rangle, |\varphi\rangle)$$

When scalar is in the first argument, we need to take its complex conjugate to take it out. That is why it is Anti-linear in its first argument.

(3) Positive $\quad SP(|\psi\rangle, |\psi\rangle \geq 0 \quad = 0$ iff $|\psi\rangle = 0$

$$u_1 \cdot u_2 = c \Rightarrow \underbrace{(x_1\ \ y_1\ \ z_1)}_{\text{row}} \underbrace{\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}}_{\text{column}} = c$$

$$c = x_1 x_2 + y_1 y_2 + z_1 z_2$$

but the addition is not defined on this hence they must be on diff vector spaces

"column" belongs to the vector space
"row" belongs to the dual space

Linear map:

A row vector maps column vector to a scalar.

$$\underbrace{(x_1, y_1, z_1)}_{\text{row}} \underbrace{\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}}_{\text{column}} = \underbrace{c}_{\text{scalar.}}$$

$\underbrace{SP(u_1, \circ)}$ maps $\underline{u_2}$ to c

Linear maps. (the objects that acts on a vector)

$$u \longleftrightarrow SP(u, \circ)$$

The set $\{ SP(u, \circ) \}$ forms a vector space: dual space.

Dual space to state space

$$SP(|\psi\rangle, |\varphi\rangle) = c \qquad c \in \mathbb{C}$$

$SP(|\psi\rangle, \circ) \implies \langle\psi|$ dual vector. "bra" "row"

$SP(|\psi\rangle, |\varphi\rangle) \Rightarrow \langle\psi|\varphi\rangle$ bracket

$|\psi\rangle \in V \quad \longrightarrow \quad \langle\psi| \in V^*$ (dual space)

- Scalar product is anti-linear in the first argument.

$$a|\psi\rangle \longrightarrow a^* \langle\psi|$$

state space		dual space.
ket	$\longleftrightarrow$	bra
$|\psi\rangle$	$\longleftrightarrow$	$\langle\psi|$
$a|\psi\rangle$	$\longleftrightarrow$	$a^*\langle\psi|$

bracket : $\langle\psi|\varphi\rangle = c$