Hermitian operators in quantum mechanics

they allow us to define physical observables, and their eigenvalues are real numbers. and the eigen state of a Hermitian operator form a basis in the state space.

$$\hat{A} = \hat{A}^+$$ an operator which is equal to its adjoint.

$$\hat{A}|\psi\rangle = |\psi'\rangle \longleftrightarrow \langle\psi|A^+ = \langle\psi'|$$
$$\longleftrightarrow \langle\psi|\hat{A} = \langle\psi'|$$

(1) Eigenvalues of a Hermitian operator are real numbers since they represent physical observable and hence should have real eigenvalues.

(2) Eigenstates of an Hermitian operator can be chosen to form an orthonormal set
we can always write an arbitrary quantum state in the representation given by the eigenstate of a physical observable.

$$A|\psi\rangle = \lambda|\psi\rangle \sim \text{Eigenvalue Equation.}$$

$$A|\psi\rangle = \lambda|\psi\rangle \Longrightarrow \langle\psi|A|\psi\rangle = \lambda\langle\psi|\psi\rangle$$
$$\langle\psi| \rightarrow$$

$$\langle\psi|\hat{A}^+ = \lambda^*\langle\psi|$$

we just need to show that $\lambda^* = \lambda$ and hence eigenvalues are $\mathbb{R}$.

$$\langle\psi|\hat{A} = \lambda^*\langle\psi|$$

$$\langle\psi|\hat{A}|\psi\rangle = \underbrace{\lambda^* \langle\psi|\psi\rangle}_{} \quad \in |\psi\rangle$$

$\lambda = \lambda^*$

$\lambda \in \mathbb{R}$

$$\lambda \underbrace{\langle\psi|\psi\rangle}_{\geq 0} = \lambda^* \langle\psi|\psi\rangle$$

$= 0$ iff $|\psi\rangle = 0$

Eigenstates

$\hat{A}|\psi\rangle = \lambda|\psi\rangle$ $\qquad$ $\hat{A}|\varphi\rangle = \mu|\varphi\rangle$

$\lambda \neq \mu$

$$\hat{A}|\psi\rangle = \lambda|\psi\rangle \xRightarrow{\langle\varphi| \in} \boxed{\langle\varphi|\hat{A}|\psi\rangle = \lambda\langle\varphi|\psi\rangle}$$

$A|\varphi\rangle = \mu|\varphi\rangle \Rightarrow \langle\varphi|A = \mu\langle\varphi|$

$\hat{A} = A^+$ $\qquad$ $\mathbb{R}$

$$\langle\varphi|A| = \mu\langle\varphi| \xrightarrow[\in |\psi\rangle]{} \boxed{\langle\varphi|\hat{A}|\psi\rangle = \mu\langle\varphi|\psi\rangle}$$

so $\lambda\langle\varphi|\psi\rangle = \mu\langle\varphi|\psi\rangle$

$$\underbrace{(\lambda-\mu)}_{\neq 0}\underbrace{\langle\varphi|\psi\rangle}_{=0} = 0$$

$\langle\varphi|\psi\rangle = 0$

$\therefore$ eigenstate of Hermitian operator, corresponding to different eigenvalues form an orthonormal pair.

Degenerate Eigenvalues

what happen to the eigenstates then?

$$\hat{A} = A^{\dagger}$$

(1) $\hat{A}|\psi^i\rangle = \lambda|\psi^i\rangle \qquad i = 1, 2, \ldots, n$

the n- eigenstates that share the same eigenvalues λ.

$|\varphi\rangle = \sum c_i |\psi^i\rangle$ linear combination of these also have the same eigenvalue

$\hat{A}|\varphi\rangle = \lambda|\varphi\rangle$

for an n-fold degenerate eigenvalue, any ket that lives in the n-dim subspace of state space spanned by the eigenkets $|\psi^i\rangle$ is also a valid eigen state.

(2) For n- fold degenerate eigenvalues, there are n linearly independent eigenstates.

N ~ dim state space

$\Downarrow$

N ~ linearly independent eigenstates (may not be mutually orthogonal)

How to make all of them mutually orthogonal?

Gram- Schmidtt orthonormalization

$$\{|\psi^i\rangle\} \longrightarrow \{|\varphi^i\rangle\}$$

set of linearly independent but not necessarily orthogonal $\rightarrow$ mutually orthonormal states $\qquad \langle\varphi^i|\varphi^j\rangle = \delta_{ij}$

(1) $|\varphi^1\rangle = \dfrac{|\psi^1\rangle}{\sqrt{\ldots}}$ make sure the state is

$\sqrt{\langle \varphi^1 | \varphi^1 \rangle} \rightarrow$ to make sure the state is normalized.

② $|\chi^2\rangle = |\psi^2\rangle + \alpha|\varphi^1\rangle$ → we want $|\chi^2\rangle$ to be orthogonal to $|\varphi^1\rangle$

$\langle \varphi^1 | \chi^2 \rangle = 0$

$\Rightarrow \langle \varphi^1 | \psi^2 \rangle + \alpha \langle \varphi^1 | \varphi^1 \rangle = 0$

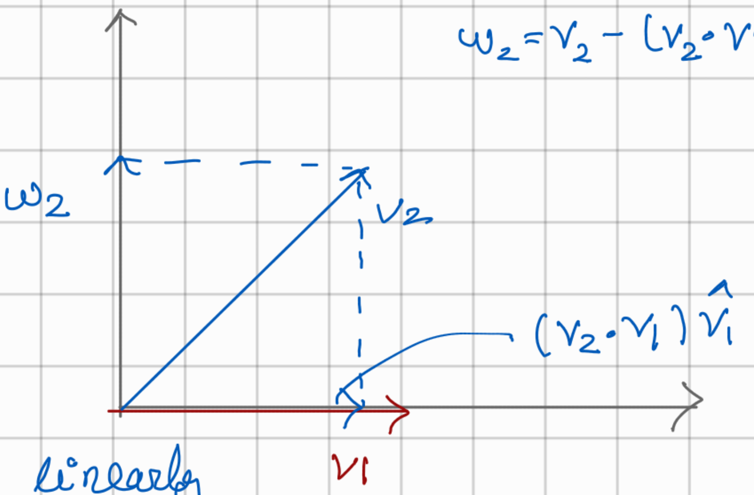
if we chose $\alpha = -\langle \varphi^1 | \psi^2 \rangle$

Replace this in ②

$|\chi^2\rangle = |\psi^2\rangle - |\varphi^1\rangle\langle \varphi^1 | \psi^2 \rangle$

$|\varphi^2\rangle = \dfrac{|\chi^2\rangle}{\sqrt{\langle \chi^2 | \chi^2 \rangle}}$ $\quad |\varphi^2\rangle$ is parallel to $|\chi^2\rangle$

$w_2 = v_2 - (v_2 \cdot v_1)\hat{v}_1$



Suppose v_1 and v_2 are linearly independent, but not orthogonal. So keeping v_1 as reference, we want to make a state which is orthonormal to it.

Repeat the same procedure.

Matrix formulation

~ transpose conjugate matrix

$\hat{A} = A_{ij}$ $\qquad$ $\hat{A}^{+} = A^{*}_{ji}$

$$A_{ij} = A^{*}_{ji}$$

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots \\ A_{21} & A_{22} & A_{23} & \cdots \\ A_{31} & A_{32} & A_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$i = j \qquad A_{ii} = A^{*}_{ii} \Rightarrow A_{ii} \in \mathbb{R}$

$i \neq j \Rightarrow A_{ij} = A^{*}_{ji}$

$$\Downarrow$$

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots \\ A_{12}^{*} & A_{22} & A_{23} & \cdots \\ A_{13}^{*} & A_{23}^{*} & A_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$\hat{A}\,|\varphi_m\rangle = \lambda_m |\varphi_m\rangle, \quad \langle \varphi_m | \varphi_n \rangle = \delta_{mn}$

$A_{mn} = \langle \varphi_m | \underbrace{\hat{A} | \varphi_n\rangle}_{\lambda_n |\varphi_n\rangle} \Rightarrow \lambda_n \langle \varphi_m | \varphi_n \rangle \Rightarrow \lambda_n \delta_{mn}$

matrix is diagonal
and the diagonal values are just the eigenvalues

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 & \cdots \\ & \lambda & 0 & 0 & \cdots \end{pmatrix}$$

$\lambda \in \mathbb{R}$

$$\begin{pmatrix} 0 & \lambda_2 & 0 & 0 & \cdots \\ 0 & 0 & \lambda_3 & 0 & \cdots \\ 0 & 0 & \cdot & \lambda_4 & \\ \vdots & \vdots & \vdots & & \end{pmatrix}$$

$\lambda_m \in \mathbb{R}$.

the matrix associate with an operator $\hat{A}$ is diagonal when written in the basis spanned by the eigenstate of $\hat{A}$, and the diagonal entries are the eigenvalues of the operator.