Operators in quantum Mechanics

An operator is a mathematical object that allows us to define physical properties in Quantum Mechanics.

Position $\rightarrow \hat{x}$
Momentum $\rightarrow \hat{p}$
Energy $\rightarrow \hat{H}$

Postulate II: A physical quantity A is described by an operator $\hat{A}$ acting on the state space V, and this operator is an observable.

$|\psi\rangle, |\psi'\rangle \in V$

$$\hat{A}|\psi\rangle = |\psi'\rangle$$

$$\hat{A}\left(a_1|\psi_1\rangle + a_2|\psi_2\rangle\right) = a_1\hat{A}|\psi_1\rangle + a_2\hat{A}|\psi_2\rangle$$

Linear operator.

Operator: addition and multiplication

Addition

~ Associative $\qquad \hat{A}+(\hat{B}+\hat{C}) = (\hat{A}+\hat{B})+\hat{C}$

~ Commutative $\qquad \hat{A}+\hat{B} = \hat{B}+\hat{A}$

Multiplication

$$(\hat{A}\hat{B})|\psi\rangle = \left(\hat{A}\,\underbrace{(\hat{B}|\psi\rangle)}_{|\psi'\rangle}\right) \Rightarrow \hat{A}|\psi'\rangle$$

~ Associative $\qquad \hat{A}(\hat{B}\hat{C}) = (\hat{A}\hat{B})\hat{C}$

~ not commutative $\qquad \hat{A}\hat{B} = \hat{B}\hat{A}$

they generally do not commute

commutator $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$

they are very important

For example, two operators that do not commute are associated with properties that cannot be measured simultaneously. Eg. $[\hat{x},\hat{p}] \neq 0$

Operator: Matrix Element

$$\langle\psi| \underbrace{(\hat{A}|\varphi\rangle)}_{|\varphi'\rangle} = \langle\psi|\varphi'\rangle = c \in \mathbb{C}$$

$\langle\psi|\hat{A}|\varphi\rangle$ this is a matrix element.

we know how an operator acts on a ket, but how does it act on a bra?

Adjoint Operator: to every ket there corresponds a bra.

$$|\psi\rangle \longleftrightarrow \langle\psi|$$

$$|\psi'\rangle = \hat{A}|\psi\rangle \longleftrightarrow \langle\psi'|$$

but how this $\langle\psi'|$ is related to $|\psi\rangle$

$$\langle\psi'| = \langle\psi|\hat{A}^{\dagger}$$

↳ Adjoint operator.

Adjoint operator is linear,

$$\langle\psi| = a_1\langle\psi_1| + a_2\langle\psi_2| \longleftrightarrow |\psi\rangle = a_1^*|\psi_1\rangle + a_2^*|\psi_2\rangle$$

because the relation b/w kets and bras is anti-linear.

so we need to take complex conjugate of any scalar.

$$\underbrace{\hat{A}|\psi\rangle}_{|\psi'\rangle} = \hat{A}\left(a_1^*|\psi_1\rangle + a_2^*|\psi_2\rangle\right) = a_1^*\underbrace{\hat{A}|\psi_1\rangle}_{|\psi_1'\rangle} + a_2^*\underbrace{\hat{A}|\psi_2\rangle}_{|\psi_2'\rangle}$$

$$|\psi'\rangle = a_1^*|\psi_1'\rangle + a_2^*|\psi_2'\rangle$$

$$|\psi'\rangle \longleftrightarrow \langle\psi'|$$

$$a_1^*|\psi_1'\rangle + a_2^*|\psi_2'\rangle \longleftrightarrow a_1\langle\psi_1'| + a_2\langle\psi_2'|$$

$$\Rightarrow a_1\langle\psi_1|\hat{A}^+ + a_2\langle\psi_2|\hat{A}^+$$

Hence a linear operator.

$$\langle\psi'| = \langle\psi|A^+$$

Revisit Postulate II

$$\hat{A} = \hat{A}^+ \quad \text{(Hermitian operator)}$$

the operators which are equal to their adjoint.

$$\hat{A}^{-1} = \hat{A}^+ \quad \text{(unitary operator)}$$

the operators where the inverse is equal to the adjoint.

Properties of operators and their adjoints

$$\langle\psi|\hat{A}^+|\varphi\rangle = \langle\varphi|\hat{A}|\psi\rangle^*$$

$$(\hat{A}^+)^+ = \hat{A}$$

$$(a(\hat{A}))^+ = a^*\hat{A}^+$$

$$(\hat{A}+\hat{B})^+ = \hat{A}^+ + \hat{B}^+$$

$$(\hat{A}\hat{B})^+ = \hat{B}^+\hat{A}^+$$

Operator: outer product as an operator

$$\left.\begin{array}{l} \langle \psi | \in V^* \\ | \varphi \rangle \in V \end{array}\right\} \text{ inner (scalar) product} \quad \langle \psi | \varphi \rangle = c \in \mathbb{C}$$

~ Does $|\varphi\rangle\langle\psi|$ have a meaning? outer product

$$(|\varphi\rangle\langle\psi|)|\chi\rangle \Rightarrow |\varphi\rangle \underbrace{(\langle\psi|\chi\rangle)}_{\text{scalar}} = a|\varphi\rangle$$

So, what is this outer product doing? Operator

since operator is an object that acts on a ket and give you another ket.

Adjoint operator in quantum mechanics

Adjoint operator help us to connect state space with dual space.

$$|\psi'\rangle = \hat{A}|\psi\rangle \longleftrightarrow \langle\psi'| = \langle\psi|\hat{A}^{+}$$

$$\langle\psi|\hat{A}^{+}|\varphi\rangle \longleftrightarrow \langle\varphi|\hat{A}|\psi\rangle^{*}$$