Quantum Computation and Quantum Information

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Problem 2.2 (Properties of the Schmidt number)

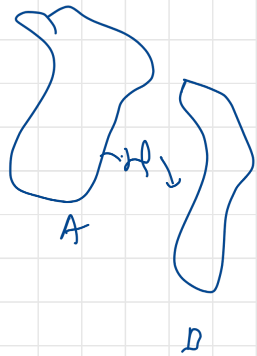
$|\psi\rangle$ is a pure state of composite system A, B

(1) Prove that Schmidt number of $|\psi\rangle$ = rank of the reduced density matrix $\rho_A \equiv \mathrm{tr}_B(|\psi\rangle\langle\psi|)$

(Rank of Hermitian operator = dim of its support)

Given a composite system A, B and a pure quantum state $|\psi\rangle$ the schmidt decomposition of it is

$$|\psi\rangle = \sum_{i=1}^{r} \sqrt{p_i}\, |a_i\rangle \otimes |b_i\rangle$$



Here r = Schmidt rank / Schmidt number.

p_i = non-negative coefficients

$|a_i\rangle, |b_i\rangle$ = orthogonal states of A and B

This schmidt number 'r' tell us how many entangled pairs are needed to optimally describe the quantum state $|\psi\rangle$.

Now the reduced density matrix ρ_A obtained by tracing out the system B:

$$\rho_A = \mathrm{tr}_B(|\psi\rangle\langle\psi|)$$

The density matrix ρ_A acts on subsystem A and can be expressed as

$$\rho_A = \sum_{i,j} \sqrt{p_i p_j}\, \langle b_i|\psi\rangle\langle\psi|b_j\rangle \otimes |a_i\rangle\langle a_j|$$

Now $\sum p_i = 1$, and the states $|a_i\rangle$ are orthogonal, that means

$$|a_i\rangle\langle a_j| \neq 0 \text{ iff } i=j$$

Similarly $\langle b_i|\psi\rangle\langle\psi|b_j\rangle \neq 0$ iff $i=j$

$\therefore$ the rank of ρ_A = number of non-zero terms in summation
i.e r

$$\text{rank}(\rho_A) = r$$

Also, the support of ρ_A is the space spanned by the non-zero eigenvalues. Since $\text{rank}(\rho_A) = r$, the $\dim(\rho_A) = r$

2) Suppose $|\psi\rangle = \sum_j |\alpha_j\rangle|\beta_j\rangle$ is representation for $|\psi\rangle$, where $|\alpha_j\rangle$ and $|\beta_j\rangle$ are un-normalized states for A and B.

Prove that the number of terms in such a decomposition $\geq$ $\text{Sch}(\psi)$

Now $|\psi\rangle = \sum_j |\alpha_j\rangle|\beta_j\rangle$

$$\text{Sch}(|\psi\rangle) = \sum_{i=1}^{r} \sqrt{p_i}\, |a_i\rangle \otimes |b_i\rangle$$

Now consider the normalized states

$$|\tilde{\alpha}_j\rangle = \frac{|\alpha_j\rangle}{\|\alpha_j\|} \qquad |\tilde{\beta}_j\rangle = \frac{|\beta_j\rangle}{\|\beta_j\|}$$

We can now express $|\psi\rangle$ in terms of these normalized states

$$|\psi\rangle = \sum_j \|\alpha_j\| \|\beta_j\| |\tilde{\alpha}_j\rangle |\tilde{\beta}_j\rangle$$

and schmidt decomposition is

$$|\psi\rangle = \sum_{i=1}^{r} \sqrt{p_i}\, |a_i\rangle \otimes |b_i\rangle$$

Each term in the schmidt decomposition corresponds to a specific pair of normalized stats.

The representation involving normalized states include all possible pairs of orthogonal state that contribute to Schmidt decomposition.

$\therefore$ the number of terms in normalized states representation is $\geq$ the Schmidt number of $|\psi\rangle$.

3) Suppose $|\psi\rangle = \alpha|\varphi\rangle + \beta|\gamma\rangle$. Prove that

$$Sch(\psi) \geq |Sch(\varphi) - Sch(\gamma)|$$

Now

$$|\varphi\rangle = \sum_{i=1}^{Sch(\varphi)} \sqrt{p_i}\, |a_i\rangle \otimes |b_i\rangle \qquad |\gamma\rangle = \sum_{j=1}^{Sch(\gamma)} \sqrt{q_j}\, |c_j\rangle \otimes |d_j\rangle$$

now

$$|\psi\rangle = \sum_{i=1}^{Sch(\varphi)} \sqrt{p_i}\, \alpha |a_i\rangle \otimes |b_i\rangle + \sum_{j=1}^{Sch(\gamma)} \sqrt{q_j}\, \gamma |c_j\rangle \otimes |d_j\rangle$$

Now the Schmidt decomposition of $|\psi\rangle \Rightarrow Sch(\psi) = r_\psi$

$Sch(\varphi) = r_\varphi \quad Sch(\gamma) = r_\gamma$

The schmidt number r_ψ for $|\psi\rangle$ is the total number of non-zero terms in the combined expression. Using Δ inequality for abs. value

$$|r_\varphi - r_\gamma| \leq r_\varphi + r_\gamma$$

Now since r_ψ is the total number of terms

$$r_\psi \geq | r_\phi - r_\gamma | \geq 0$$ Hence

$$Sch(\psi) \geq | Sch(\phi) - Sch(\gamma) |$$

Prob 2-3) Tsirelson's Inequality

$Q = \vec{q}\cdot\vec{\sigma}$ $R = \vec{r}\cdot\vec{\sigma}$ $S = \vec{s}\cdot\vec{\sigma}$ $T = \vec{t}\cdot\vec{\sigma}$

$\vec{q}, \vec{r}, \vec{s}$ and $\vec{t}$ are real unit vectors, we have to show

$$(Q\otimes S + R\otimes S + R\otimes T - Q\otimes T)^2 = 4I + [Q,R]\otimes[S,T]$$

(Brace yourself)

starting from

$$(Q\otimes S + R\otimes S + R\otimes T - Q\otimes T)^2 = 4I + [Q,R]\otimes[S,T]$$

which we can rewrite as

$$[(Q+R)\otimes S + (R-Q)\otimes T]^2 = 4I + (QR - RQ)\otimes(ST - TS)$$

Expanding the LHS:

$$(Q+R)^2\otimes S^2 + (R-Q)^2\otimes T^2 + (R+Q)(R-Q)\otimes ST + (R-Q)(R+Q)\otimes TS$$

$$(Q+R)^2\otimes S^2 + (R-Q)^2\otimes T^2 + (R^2 - RQ + QR - Q^2)\otimes ST + (R^2 + RQ - QR - Q^2)\otimes TS$$

Taking out RQ and QR terms from the last two sets we get

$$(Q+R)^2\otimes S^2 + (R-Q)^2\otimes T^2 + (R^2 - Q^2)\otimes ST + (R^2 - Q^2)\otimes TS + (QR - RQ)\otimes ST - (QR - RQ)\otimes TS$$

$$\Rightarrow (Q+R)^2\otimes S^2 + (R-Q)^2\otimes T^2 + (R^2 - Q^2)\otimes ST + (R^2 - Q^2)\otimes TS + (QR - RQ)\otimes (ST - TS)$$

But the last term in the expression is also present in the RHS of our main equation. Removing this from both sides, we now have to prove

$$(Q+R)^2 \otimes S^2 + (R-Q)^2 \otimes T^2 + (R^2-Q^2) \otimes ST + (R^2-Q^2) \otimes TS = 4I$$

Since all Pauli matrices are Hermitian we know that Q, R, S and T are also hermitian. Hence we can make use of the spectral decomposition to show that these matrices are diagonalizable. Q has eigenvalues ± 1, thus

$$Q = |q_1\rangle\langle q_1| - |q_2\rangle\langle q_2|$$ $|q_1\rangle$ and $|q_2\rangle$ are orthonormal basis

$$Q^2 = |q_1\rangle\langle q_1| + |q_2\rangle\langle q_2| = I$$ (Hence $Q^2, S^2, R^2, T^2 = I$)

By the completeness relation

$$(2I + RQ + QR) \otimes I + (2I - RQ - QR) \otimes I + (I - I) \otimes ST + (I - I) \otimes TS = 4I$$

$$(4I) \otimes I + (\underline{RQ} + \underline{\underline{QR}} - \underline{RQ} - \underline{\underline{QR}}) \otimes I = 4$$

$$4I = 4I.$$

Using this identity we prove

$$\langle Q \otimes S\rangle + \langle R \otimes S\rangle + \langle R \otimes T\rangle - \langle Q \otimes T\rangle \leq 2\sqrt{2}.$$

i.e the CHSH inequality

This upper limit implies that no physical system can exceed this upper bound.

The upper bounds that entangled quantum states do not violate are called Tsirelson's bound.

We have:

$$Q = \vec{q} \cdot \vec{\sigma} \qquad R = \vec{r} \cdot \vec{\sigma} \qquad S = \vec{s} \cdot \vec{\sigma} \qquad T = \vec{t} \cdot \vec{\sigma}$$

To start we define four projections P_0, P_1, P_2, P_3

$$Q = 2P_0 - \mathbb{1} \qquad R = 2P_1 - \mathbb{1} \qquad S = 2P_2 - \mathbb{1} \qquad T = 2P_3 - \mathbb{1}$$

we know the eigenvalues for the operators are $\in \{-1, 1\}$

Now we define an operator which combines

$$C = Q \otimes (S+T) + R \otimes (T-S)$$

Let's just use $Q = A$, $S = B$, $T = B'$, $R = A'$

So, we get

$$C = A \otimes (B + B') + A' \otimes (B' - B)$$

squaring it

$$C^2 = (A \otimes (B+B'))(A \otimes (B+B')) + (A \otimes (B+B'))(A' \otimes (B'-B)) +$$
$$(A' \otimes (B'-B))(A \otimes (B+B')) + (A' \otimes (B'-B))(A' \otimes (B'-B))$$

$$\Rightarrow A^2 \otimes (B^2 + B'^2 + BB' + B'B) + A'^2 \otimes (B^2 + B'^2 - BB' - B'B) + (AA') \otimes (B'^2 + BB' - B'B - B^2)$$
$$+ (A'A) \otimes (B^2 - BB' + B'B - B'^2)$$

$$\Rightarrow \mathbb{1} \otimes (\mathbb{1} + \mathbb{1} + BB' + B'B) + \mathbb{1} \otimes (\mathbb{1} + \mathbb{1} - BB' - B'B) + (A'A) \otimes (\mathbb{1} + BB'$$
$$- B'B - \mathbb{1}) + (A'A) \otimes (\mathbb{1} - BB' + B'B - \mathbb{1})$$

$$\Rightarrow \mathbb{1} \otimes (4\mathbb{1}) + (AA') \otimes (BB' - B'B) - (A'A) \otimes (BB' - B'B)$$

$$\Rightarrow 4\mathbb{1} + (AA' - A'A) \otimes (BB' - B'B)$$

where we used that $A^2 = (2P - \mathbb{1})^2 = 4P^2 - 4P + \mathbb{1} = \mathbb{1}$ $\forall$ of the ops.

Now putting the explicit form back

$$C^2 = 4\mathbb{1} + [A, A'] \otimes [B, B'] = 4\mathbb{1} + 16[P_0, P_2] \otimes [P_1, P_3]$$

if we take the norm of C^2, we get

$$\|C^2\| = \|4\mathbb{1} + [A,A'] \otimes [B,B']\| \leq 4 + \|[A,A']\| \, \|[B,B']\|$$

(we used many norm relations to get here)

we will only look at what happens with $[A,A']$ when we take the norm,

$$\|[A,A']\| \leq \|AA'\| + \|A'A\| \leq \|A\|\|A'\| + \|A'\| \, \|A\|$$

$$\Rightarrow \sup_{\|\vec{x}\|=1} \left(\|A\vec{x}\| \|A'\vec{x}\| + \|A'\vec{x}\| \, \|A\vec{x}\| \right) = 2$$

We also made use of the fact that any vector can be expressed as linear combinations of eigenvectors of A or A'. thus $\|A\vec{x}\| = 1$

Inserting everything back

$$\|C^2\| \leq (4 + 2 \cdot 2) = 8$$

Now making use of $(\Delta C)^2$

$$(\Delta C)^2 = \sqrt{\langle C - \langle C \rangle)^2}$$ (always non negative if the eigenvalues $\in \mathbb{R}$)

$$\langle C^2 \rangle \geq \langle C \rangle^2$$

with all the above we get

$$S = \langle C \rangle \leq \langle \|C^2\| \rangle = 2\sqrt{2}$$

Exercise 4.10 $^{4.1]}$ Suppose U is a unitary operation on a single qubit. There exist real numbers, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ s.t

$$U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta)$$

Now since U is unitary, the rows and columns of it are orthonorm.

$$U = \begin{bmatrix} e^{i(\alpha-\beta/2-\delta/2)} \cos\frac{\gamma}{2} & -e^{i(\alpha-\beta/2+\delta/2)} \sin\frac{\gamma}{2} \\ e^{i(\alpha+\beta/2-\delta/2)} \sin\frac{\gamma}{2} & e^{i(\alpha+\beta/2+\delta/2)} \cos\frac{\gamma}{2} \end{bmatrix}$$

Now doing the same thing but making use of R_x instead of R_z

Any single qubit gate U can be decomposed as

$$U = e^{i\theta_0} e^{-i\theta_1 \sigma_z/2} e^{-i\theta_2 \sigma_y/2} e^{-i\theta_3 \sigma_z/2}$$

$$\Rightarrow e^{i\theta_0} \begin{bmatrix} e^{i\theta_1/2} & 0 \\ 0 & e^{i\theta_1/2} \end{bmatrix} \begin{bmatrix} \cos\frac{\theta_2}{2} & -\sin\frac{\theta_2}{2} \\ \sin\frac{\theta_2}{2} & \cos\frac{\theta_2}{2} \end{bmatrix} \begin{bmatrix} e^{-i\theta_3/2} & 0 \\ 0 & e^{i\theta_3} \end{bmatrix}$$

where $\sigma_1 = \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad \sigma_2 = \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$\sigma_3 = \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ are the Pauli Matrices also $R_x(\theta) = e^{-i\theta\sigma_x/2} = \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} X$

$R_y(\theta) = e^{-i\theta\sigma_y/2} = \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} Y$

$R_z(\theta) = e^{-i\theta\sigma_z/2} = \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} Z$

$$\begin{bmatrix} \cos\theta/2 & -i\sin\theta/2 \\ -i\sin\theta/2 & \cos\theta/2 \end{bmatrix} \qquad \begin{bmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{bmatrix} \qquad \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$$

R_x $\qquad$ R_y $\qquad$ R_z

also make sure that:

$$R_{\hat{n}}(\theta) = e^{-i\theta\,\hat{n}\cdot\vec{\sigma}/2}$$

$$= \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2}\left(n_x X + n_y Y + n_z Z\right)$$

Hence arbitrary single qubit unitary operations can be written in the form $U = e^{i\alpha} R_{\hat{n}}(\theta)$

so the X-Y rotation decomposition become

$$U = e^{i\alpha} R_x(\beta) R_y(\gamma) R_x(\delta)$$

in fact, for non-parallel unit vectors $\hat{m}$ and $\hat{n}$ in 3-D

$$U = e^{i\alpha} R_{\hat{n}}(\beta) R_{\hat{m}}(\gamma) R_{\hat{n}}(\delta)$$

also one can def:

$ABC = I$ and $U = e^{i\alpha} AXBXC$

$A \equiv R_z(\beta) R_y(\gamma/2)$ $\quad$ $B \equiv R_y(-\gamma/2) R_z(-(\delta+\beta)/2)$, $C \equiv R_z((\delta-\beta)/2)$

4.15) composition of single qubit operations

i) Representation of Individual Rotations

$$R(\beta_1, n_1) = \cos\left(\frac{\beta_1}{2}\right) I - i\sin\left(\frac{\beta_1}{2}\right)\left(n_{1x}\sigma_x + n_{1y}\sigma_y + n_{1z}\sigma_z\right)$$

$$R(\beta_2, n_2) = \cos\left(\frac{\beta_2}{2}\right) I - i\sin\left(\frac{\beta_2}{2}\right)\left(n_{2x}\sigma_x + n_{2y}\sigma_y + n_{2z}\sigma_z\right)$$

ii) composition of Rotation

$$R_{12} = R(\beta_2, n_2)\, R(\beta_1, n_1)$$

getting the result back.

$$R_{12} = \cos\left(\frac{\beta_{12}}{2}\right) I - i\sin\left(\frac{\beta_{12}}{2}\right)\left(n_{12x}\sigma_x + n_{12y}\sigma_y + n_{12z}\sigma_z\right)$$

Equating $\mathbb{R}$ and $\mathbb{C}$ part using trig identities

$$n_{12x} = \sin\left(\frac{\beta_1}{2}\right)\cos\left(\frac{\beta_2}{2}\right) n_{1x} + \cos\left(\frac{\beta_1}{2}\right)\sin\left(\frac{\beta_2}{2}\right) n_{2x} - \sin\left(\frac{\beta_1}{2}\right)\sin\left(\frac{\beta_2}{2}\right)\left(n_{2y} n_{1z} - n_{1y} n_{2z}\right)$$

$$n_{12y} = \sin\left(\frac{\beta_1}{2}\right)\cos\left(\frac{\beta_1}{2}\right) n_{1y} + \cos\left(\frac{\beta_1}{2}\right)\sin\left(\frac{\beta_1}{2}\right) n_{2y} + \sin\left(\frac{\beta_1}{2}\right)\sin\left(\frac{\beta_2}{2}\right)\left(n_{2x} n_{12} - n_{1x} n_{2x}\right)$$

$$n_{12x} = \sin\left(\frac{\beta_1}{2}\right)\cos\left(\frac{\beta_2}{2}\right) n_{1z} + \cos\left(\frac{\beta_1}{2}\right)\sin\left(\frac{\beta_2}{2}\right) n_{2z} - \sin\left(\frac{\beta_1}{2}\right)\sin\left(\frac{\beta_2}{2}\right)\left(n_{2y} n_{1x} - n_{1y} n_{2z}\right)$$

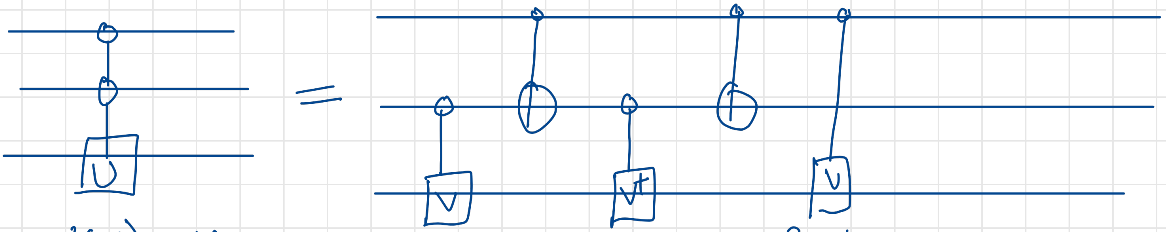
Comparing these with desired result

$$\cos\left(\frac{\beta_{12}}{2}\right) = \cos\left(\frac{\beta_1}{2}\right)\cos\left(\frac{\beta_2}{2}\right) + \sin\left(\frac{\beta_1}{2}\right)\sin\left(\frac{\beta_2}{2}\right) n_1 \cdot n_2,$$

$$\sin\left(\frac{\beta_{12}}{2}\right) n_{12} = \sin\left(\frac{\beta_1}{2}\right)\cos\left(\frac{\beta_2}{2}\right) n_1 + \cos\left(\frac{\beta_1}{2}\right)\sin\left(\frac{\beta_2}{2}\right) n_2 - \sin\left(\frac{\beta_1}{2}\right)\sin\left(\frac{\beta_2}{2}\right)(n_2 \times n_1)$$

4.22) $C^2(U)$ gate

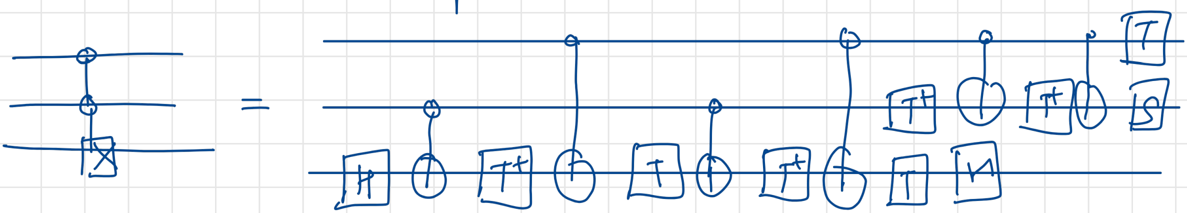
We know
$$C^n(U)|x_1 x_2 \ldots x_n\rangle|\psi\rangle = |x_1, x_2 \ldots x_n\rangle U^{x_1 x_2 \ldots x_n}|\psi\rangle$$
where $x_1, x_2 \ldots x_n$ in the exponent of U means the product of the bits $x_1, x_2 \ldots x_n$. i.e the operator U is applied to the last k qubits if the first n qubits are all equal to one.



$C^2(U)$ gate

V is a unitary s.t $V^2 = U$

Example



Step 1: Controlled -U gate

1. Apply H to the control qubit
$$H|c\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)_c$$

2. Apply the CNOT gate:
$$CNOT_{c,t} \cdot \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)_c \otimes |0\rangle_t = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

3. Apply the target Unitary U
$$U \cdot \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|0\rangle \otimes U|0\rangle + |1\rangle \otimes U|1\rangle)$$

4. Apply the inverse of CNOT

$$CNOT^{-1}_{c,t} \cdot \frac{1}{\sqrt{2}}\left(|0\rangle \otimes U|0\rangle + |1\rangle \otimes U|1\rangle\right) = \frac{1}{\sqrt{2}}\left(|0\rangle \otimes U|0\rangle + |1\rangle \otimes U|1\rangle\right)$$

Step 2: Toffoli gate (as shown in the figure above)