Lecture 2: Quantum Computation Review

Nature behaves in non-local manner.

I think that a particle must have a separate reality independent of measurements. That is, an electron has spin, location and so forth even when it is not being measured. I like to think the moon is there even if I am not looking at it

——— Albert Einstein

.... experiments have now shown that what bothered Einstein is not a debatable point but the observed behavior of the real world

——— N David Mermin

1) Linear Algebra

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_d \end{pmatrix} \in \mathbb{C}^d \qquad \psi_1 \in \mathbb{C}$$

ket $|\psi\rangle$

$a + ib \in \mathbb{C}$

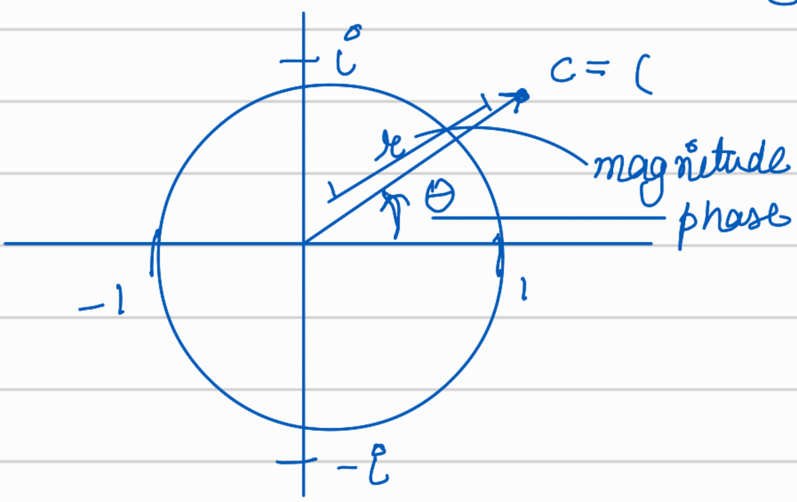
$re^{i\theta} \in \mathbb{C}$

$a, b, r, \theta \in \mathbb{R}$

complex conjugate c^*

$\Rightarrow a - ib$

$\Rightarrow re^{-i\theta}$



$c = r(\cos\theta + i\sin\theta)$ Euler's formula

$\rightarrow |c| = \sqrt{c^* c}$ or $|re^{i\theta}| = |r|\,\underbrace{|e^{i\theta}|}_{1}$

$\Rightarrow |r|\,|e^{-i\theta}e^{i\theta}|$

$\Rightarrow |r|$

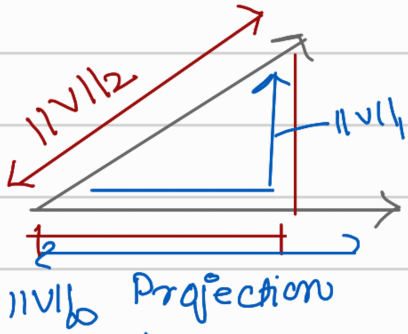
$c + c^* = 2\mathrm{Re}(c)$

$c - c^* = 2\mathrm{Im}(c)$

$cc^* = |c|^2$

"bra ψ"

$$\langle\psi| = (\psi_1^* \ \psi_2^* \ \cdots \ \psi_d^*)$$



Projection is inner product

Inner product $\underset{1\times d}{\langle\psi|} \cdot \underset{d\times 1}{|\varphi\rangle} = \langle\psi|\varphi\rangle \in \mathbb{C}$

$$\Rightarrow \sum_{i=1}^{d} \psi_i^* \varphi_i$$

Euclidean Norm $\| |\psi\rangle \|_2 = \sqrt{\langle\psi|\psi\rangle}$

length of the vector.

$$\| |\psi\rangle \|_1 = \sum_i |\psi_i| \quad \text{one norm}$$

$$\| |\psi\rangle \|_\infty = \max_i |\psi_i| \quad \text{infinity norm}$$

Orthonormal Basis Set $\{|\psi_i\rangle\} \subseteq \mathbb{C}^d$ s.t $\langle\psi_i|\psi_j\rangle = \delta_{ij}$ ← Kronecker delta

$$\delta_{ij} = \begin{cases} 1, & \text{if } i=j \\ 0 & \text{o/w} \end{cases}$$

standard basis / computational basis

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \cdots \quad |d-1\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Any $|\psi\rangle \in \mathbb{C}^d$ as $|\psi\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle \in \mathbb{C}^d$

if $\| |\psi\rangle \|_2 = 1 \Rightarrow \sum_i |\alpha_i|^2 = 1$

normalized

Linear Maps and Matrices Map $\Phi : \mathbb{C}^d \mapsto \mathbb{C}^d$ is linear if

$$\Phi\left(\sum_i \alpha_i |\psi_i\rangle\right) = \sum_i \alpha_i \Phi(|\psi_i\rangle) \quad \text{(linearity)}$$

$\mathcal{L}(\mathcal{X},\mathcal{Y}) \rightarrow$ set of linear maps from $\mathcal{X} \mapsto \mathcal{Y}$

$\mathcal{L}(\mathcal{X}) \rightarrow \Phi : \mathcal{X} \mapsto \mathcal{X}$

matrix representation of $\Phi : \mathbb{C}^d \mapsto \mathbb{C}^d$, A_Φ is a $d \times d$ matrix

$$A = \left[\begin{array}{c|c|c|c} \Phi(|0\rangle) & \Phi(|1\rangle) & \cdots & \Phi(|d-1\rangle) \end{array}\right]_{d \times d} \in \mathbb{C}^{d \times d}$$

EXERCISE $\Phi(|0\rangle) = |1\rangle$ $\Phi(|1\rangle) = |0\rangle$

$$A_\Phi = \left(|1\rangle \,\vdots\, |0\rangle \right) \Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

X gate

$$\boxed{\begin{aligned} A|i\rangle &= i^{th} \text{ column of } A. \\ X|0\rangle &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow |1\rangle \end{aligned}}$$

Matrix Multiplication

$$AB_{(i,j)} = \sum_{k=1}^{d} A(i,k)\, B(k,j)$$

recall! matrix multiplication $AB \neq BA$ in general

(this keeps matter stable)

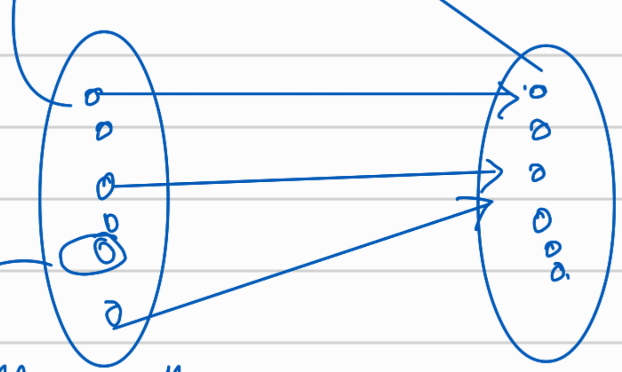
if $AB = BA$, say A and B commute.

$$[A,B] = AB - BA = 0$$

EXERCISE $XZ \neq ZX$ $(XZ = -ZX)$ They anti commute.

Spaces i) Image of A: $\mathrm{Im}(A) = \{ |\psi\rangle \in \mathbb{C}^d \mid \exists |\phi\rangle \in \mathbb{C}^d$

input output $A|\phi\rangle = |\psi\rangle \}$



ii) Null space of A

$$\mathrm{Null}(A) = \{ |\psi\rangle \in \mathbb{C}^d \mid A|\psi\rangle = 0 \}$$

$$\mathrm{rank}(A) = \dim(\mathrm{Im}(A))$$

Rank-nullity theorem

$$\mathrm{rank}(A) + \dim(\mathrm{null}(A)) = d \quad \text{(full dimension)}$$

Matrix operations a) $A^{\dagger} = (A^T)^*$ conjugate transpose, adjoint, dagger.

recall $(AB)^{\dagger} = B^{\dagger} A^{\dagger}$ (flips the order)

b) $\mathrm{Trace}(A)$ = sum of all diagonal terms $\sum_{i=1}^{d} A_{i,i}$

trace is linear

$\mathrm{Tr}: L(\mathbb{C}^d) \longmapsto \mathbb{C}$

take a matrix gives a single number.

and is cyclic

$$\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA)$$

the implication is $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ always

Outer Product

$|\psi\rangle, |\varphi\rangle \in \mathbb{C}^d \rightarrow$ outer product

$\underset{d\times 1}{|\psi\rangle}\underset{1\times d}{\langle\varphi|} \in L(\mathbb{C}^d)$

E.g $|1\rangle\langle 0| = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$|i\rangle\langle j| \rightarrow$ 1 in position (i,j), 0 otherwise

Trick $\underset{1\times d}{\langle i|}\underset{d\times d}{A}\underset{d\times 1}{|j\rangle} = \langle i| \left(\sum_{i',j'} A(i',j') |i'\rangle\langle j'| \right) |j\rangle$

std. basis are orthonormal

$\Rightarrow \sum_{i',j'} A(i',j') \underbrace{\langle i|i'\rangle}_{\delta_{ii'}} \underbrace{\langle j'|j\rangle}_{\delta_{jj'}}$

$\Rightarrow A(i,j)$ (outer product take vectors together and make matrices out of it)

EXERCISE 9: $Tr(A) = \sum_i A(i,i)$

$\sum_i \langle i|A|i\rangle$ Extracting out the diagonal enteries

Eigenvalues and Eigenvectors

A matrix $A \in L(\mathbb{C}^d)$ have eigenvector $|\psi\rangle$ if

$A|\psi\rangle = \lambda|\psi\rangle$

eigenvalues

An operator A is normal iff $AA^\dagger = A^\dagger A$ iff A has a spectral decomposition

$A = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$ eigen vectors

eigenvalues

if A is normal then the eigenvectors can be chosen as if A is normal then the eigenvectors can be chosen as orthonormal basis.

if A is normal then, (because spectral decomposition)

$$Tr(A) = Tr\left(\sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|\right)$$

(trace is linear)

$$\Rightarrow \sum_i \lambda_i Tr(|\lambda_i\rangle\langle\lambda_i|)$$

$$\Rightarrow \sum_i \lambda_i \langle\lambda_i|\lambda_i\rangle \quad \text{(trace is cyclic}$$

Trace of an operator is always the sum of its eigenvalues.

EXERCISE: If A is diagonizable (normal) the prove that

rank(A) = number of non-zero eigenvalues of A

Rank of a diagonizable matrix is equal to the number of non-zero eigenvalues

If A is diagonizable then $\exists P$ invertible matrix such that $P^{-1}AP = D$ where D is diagonal matrix having eigenvalues of A on its diagonal and each eigenvalue appears as many times as its multiplicity.

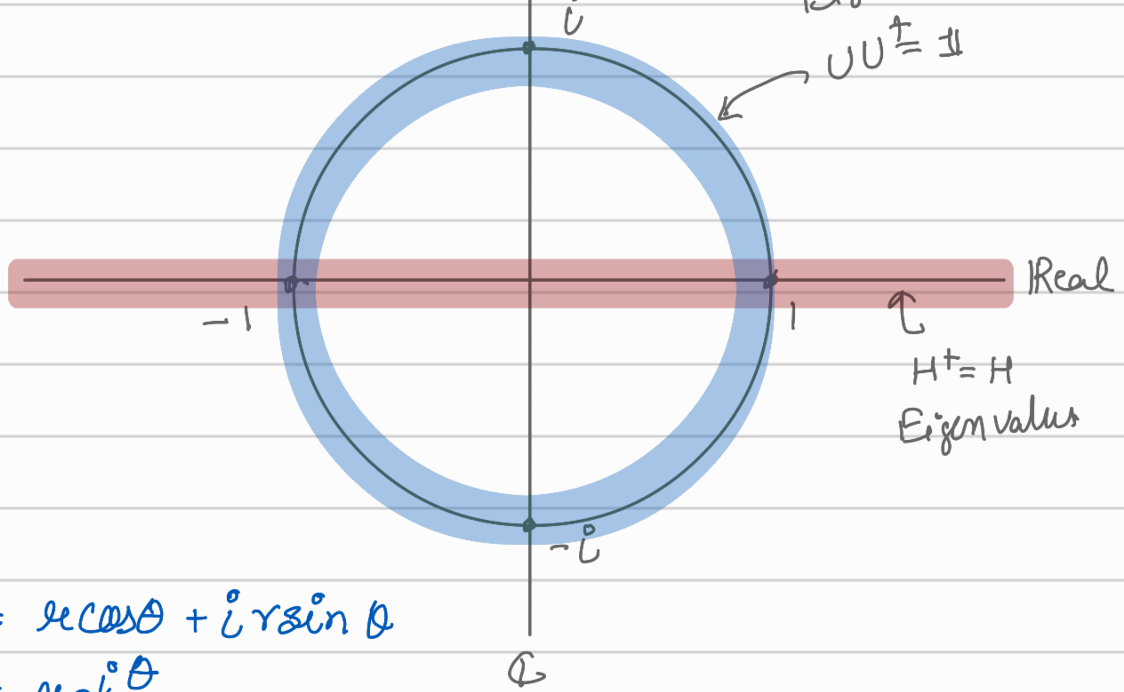
$$r(A) = r(P^{-1}AP) = r(D)$$

Important Classes of Matrices

i) unitary matrices : $U \in L(\mathbb{C}^d)$ if $UU^\dagger = \mathbb{1} = U^\dagger U$

EXERCISE: Eigenvalues of unitary operator are always on the unit circle.

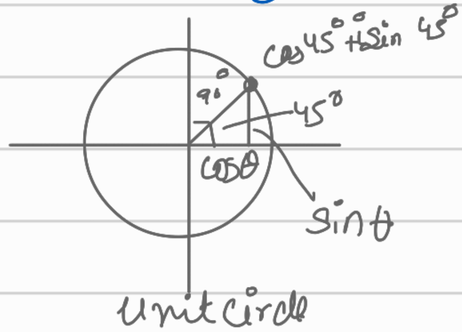
e Eigenvalues



$$x + iy = r\cos\theta + i r\sin\theta$$
$$= re^{i\theta}$$
$$r = |x+iy| = \sqrt{(x+iy)(x-iy)} \Rightarrow \sqrt{x^2+y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Let's see how can we write complex numbers using these notations.

$$\cos 45^\circ + i\sin 45^\circ \Rightarrow \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$
$$\Rightarrow e^{i\frac{\pi}{4}}$$



Unitary matrices preserves norm of a vector.

eigenvalue λ eigenvector $|\psi\rangle$

$$U|\psi\rangle = \lambda|\psi\rangle \quad \sim \text{Eigenvalue equation}$$

unitary matrices preserve norm

$$\||\psi\rangle\| = \|U|\psi\rangle\|$$
$$\||\psi\rangle\| = \|\lambda|\psi\rangle\|$$

$$\sqrt{\langle\psi|\psi\rangle} = \sqrt{(\lambda|\psi\rangle)^*(\lambda|\psi\rangle)}$$
$$= \sqrt{\langle\psi|\lambda^*\lambda|\psi\rangle}$$
$$= \sqrt{\lambda^*\lambda\langle\psi|\psi\rangle}$$
$$\sqrt{\langle\psi|\psi\rangle} \Rightarrow |\lambda|\sqrt{\langle\psi|\psi\rangle}$$

absolute value of complex number, how far is it from the center of the circle

$|\lambda| = 1$ ~ hence the eigenvalues of a unitary operator lies on the complex number

Unitary matrices are high dimensional generalization of unit complex numbers.

Set of unitary : $\mathcal{U}(\mathcal{X})$

ii) Hermitian $H \in L(\mathbb{C}^d)$ if $H = H^+$

set of Hermitian operator $\mathrm{Herm}(\mathcal{X})$

their eigen values are always $\mathbb{R}$

and they generalize real number.

iii) Positive (semi-) definite : Hermitian + non-negative or strictly positive

semi-def

definite

$\mathrm{Pos}(\mathcal{X})$

If one of your eigenvalue is 0, you're not invertible. $H \succeq 0$ semi $\quad H > 0$ definite

generalize set of non-negative and positive reals.

iv) Orthogonal Projections : $\Pi \in L(\mathbb{C}^d)$ s.t Π is Hermitian and $\Pi^2 = \Pi$

$\Pi^2 = \Pi \Rightarrow$ eigenvalues $\in \{0, 1\}$

Spectral decomp: $\Pi = \sum_i |\psi_i\rangle\langle\psi_i|$ for $\{|\psi_i\rangle\}$ orthonormal basis.

rank-1 proj : $|\psi\rangle\langle\psi|$

only one non neg eigenvalue

exponential

Some Revision on Notes

Maps $\Phi : \mathbb{C}^d \mapsto \mathbb{C}^d$ are linear, meaning

$$\sum_i \alpha_i |\psi_i\rangle \in \mathbb{C}^d \text{ that}$$

$$\Phi\left(\sum_i \alpha_i|\psi_i\rangle\right) = \sum_i \alpha_i \Phi|\psi_i\rangle$$

hence linear maps.

$\mathcal{L}(X, Y)$ they have matrix representation

$$A_\Phi = \left[\Phi|0\rangle, \Phi|1\rangle, \dots \Phi|d-1\rangle\right]$$

we define the i^{th} column of A_Φ as $\Phi|i\rangle$ for $\{|i\rangle\}$ the standard basis for $\mathbb{C}^d$.

The image of a matrix A is the set of all possible output vectors under the action of A.

$$Im(A) := \{|\psi\rangle \in \mathbb{C}^d \,|\, |\psi\rangle = A|\phi\rangle \text{ for some } \phi \in \mathbb{C}^d\}$$

The rank of A is the dimension of its image. i.e $\dim(Im(A))$

$$Null(A) := \{|\psi\rangle \in \mathbb{C}^d \,|\, A|\psi\rangle = 0\}$$

Rank- Nullity Theorem

$$\dim(Null(A)) + \dim(Im(A)) = d$$

A trace is a linear map $Tr : \mathcal{L}(\mathbb{C}^d) \longmapsto \mathbb{C}$ summing the entries on the diagonal of A.

$$Tr(A) = \sum_{i=1}^{d} A(i,i)$$

Trace is cyclic $\quad Tr(ABC) = Tr(CAB)$

Operator Functions

How to apply $f: \mathbb{R} \mapsto \mathbb{R}$ to matrices. $f(x) = \sqrt{x}$

for H being a Hermitian operator $H \in \mathrm{Herm}(\mathbb{C}^d)$ with spectral decomp $H = \sum \lambda_i |\lambda_i\rangle\langle\lambda_i|$

$f: \mathrm{Herm}(\mathbb{C}^d) \mapsto \mathrm{Herm}(\mathbb{C}^d)$

apply to scalar.

$\Rightarrow f(H) = \sum_i f(\lambda_i) |\lambda_i\rangle\langle\lambda_i|$

EXERCISE $f(x) = e^x$. Prove $f(H) = e^H := I + H + \frac{H^2}{2!} + \frac{H^3}{3!} + \dots$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

We know $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Since H is a Hermitian matrix, we can diagonalize as $H = U \Lambda U^\dagger$ where U is a unitary matrix and Λ is a diagonal matrix containing the eigenvalues of H.

$$f(H) = f(U\Lambda U^\dagger) = U f(\Lambda) U^\dagger$$

Now for $f(\Lambda) = e^\Lambda = \sum_{n=0}^{\infty} \frac{\Lambda^n}{n!}$

Since Λ is a diagonal matrix, Λ^n is also a diagonal matrix with the i-th diagonal element raised to power of n, denoted as λ_i^n, where λ_i is the i-th eigenvalue of H. $\therefore$

$$e^\Lambda = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n} \end{pmatrix}$$

Now, substitute e^{λ} back into the expression

$$f(H) = Uf(\lambda)U^{\dagger} = U\left(\sum_{n=0}^{\infty}\frac{1}{n!}\begin{pmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n}\end{pmatrix}\right)U^{\dagger}$$

By regrouping

$$f(H) = \sum_{n=0}^{\infty}\frac{1}{n!}\begin{pmatrix} Ue^{\lambda_1}U^{\dagger} & 0 & \cdots & 0 \\ 0 & Ue^{\lambda_2}U^{\dagger} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Ue^{\lambda_n}U^{\dagger}\end{pmatrix} = e^{H}$$

EXERCISE $f(x) = \sqrt{x}$, $\Rightarrow \sqrt{I} = \sqrt{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}} \Rightarrow \begin{bmatrix}\sqrt{1} & 0 \\ 0 & \sqrt{1}\end{bmatrix}$

$$\sqrt{I} \leftarrow \left\{\begin{bmatrix}1 & \\ & 1\end{bmatrix}, \begin{bmatrix}-1 & \\ & -1\end{bmatrix}, \begin{bmatrix}-1 & \\ & 1\end{bmatrix}, \begin{bmatrix}1 & \\ & -1\end{bmatrix}\right\}$$

Basic Quantum Computation

Pure State computation - d- dimensional pure state $|\psi\rangle = \sum_{i=0}^{d-1}\alpha_i|i\rangle \in \mathbb{C}^d$

$\||\psi\rangle\|_2 = 1$

d-dim = qudit

2-dim = qubit

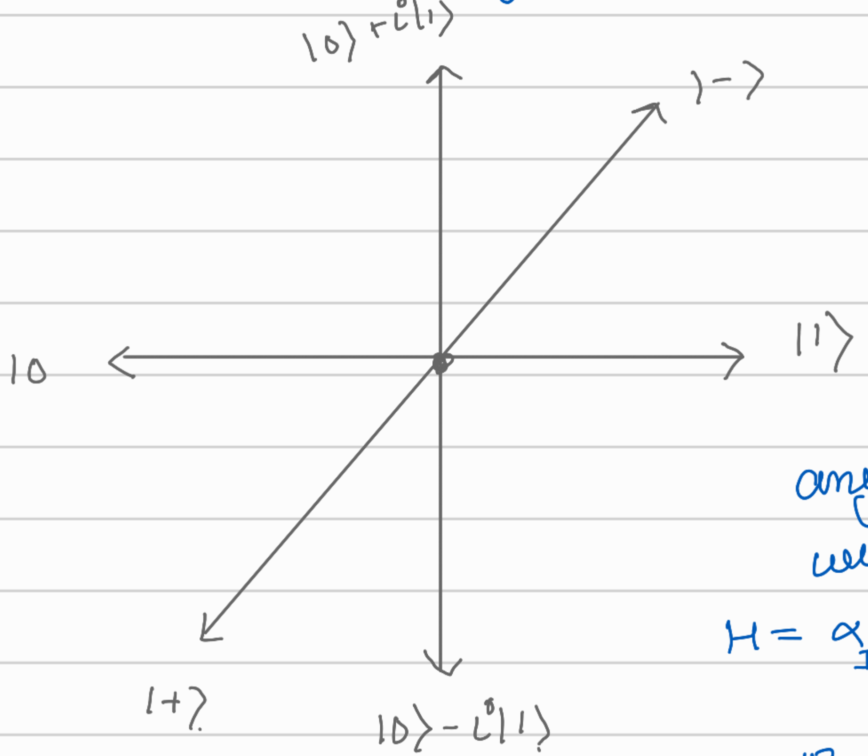
Quantum gates - set of allowable operators are $U \in \mathcal{U}$, such that $UU^{\dagger} = \mathbb{I}$. The gates are all reversible.

For d=2 , we see $X = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ $Y = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$ $Z = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$

NOT gate

Phase gate

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ gate



form a matrix basis for Hermitian Herm($\mathbb{C}^2$) i.e.

any $H \in$ Herm($\mathbb{C}^2$) can be written as

$$H = \alpha_I I + \alpha_x X + \alpha_y Y + \alpha_z Z$$

$\alpha_i \in \mathbb{R}$

Hadamard $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

introduces superposition

$H|0\rangle = |+\rangle \Rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$H|1\rangle = |-\rangle \Rightarrow \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

Relative Phase $Z|0\rangle = |0\rangle$

$Z|1\rangle = -|1\rangle$

Phase

$Z|+\rangle = \frac{1}{\sqrt{2}} Z|0\rangle + \frac{1}{\sqrt{2}} Z|1\rangle = |-\rangle$

$Z|-\rangle = |+\rangle$

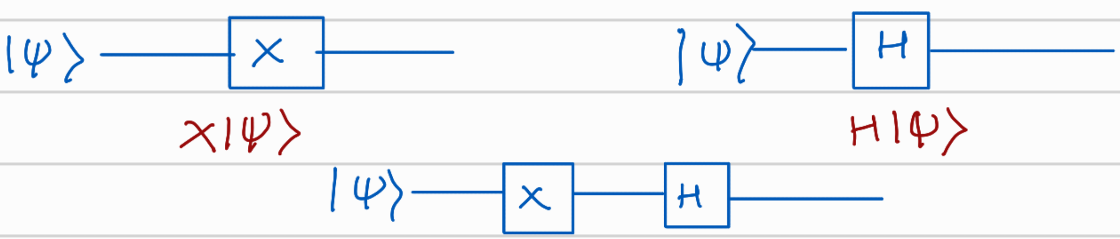
NOT gate

Global phase

$e^{i\theta}|0\rangle + e^{i\theta}|1\rangle \Rightarrow e^{i\theta}(|0\rangle + |1\rangle)$

we cannot detect global phase in measurement.

Quantum circuit model



$HX|\psi\rangle$

EXERCISE What gate does this circuit simulate?

Hint: spectral decompose X, Z

$$\underbrace{|0\rangle - \boxed{H} - \boxed{X} - \boxed{H} -}_{?}$$

Soln: $X = |0\rangle\langle 1| + |1\rangle\langle 0|$

$$H = \frac{1}{\sqrt{2}}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\right)$$

$$HXH|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\right)\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

we get

$$HXH|\psi\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle - |0\rangle\right) = -Z|\psi\rangle$$

Hence, the circuit HXH simulates a Z gate.

Composite Quantum Systems

we make use tensor product, $\otimes : \mathbb{C}^{d_1} \times \mathbb{C}^{d_2} \to \mathbb{C}^{d_1 \times d_2}$

$$\underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n \text{ copies}} \cong \mathbb{C}^{2^n}$$

formally if we have

$$\left(|\psi\rangle \otimes |\varphi\rangle\right)(i,j) = \psi_i \varphi_j$$

Eg.

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a\begin{pmatrix} c \\ d \end{pmatrix} \\ b\begin{pmatrix} c \\ d \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bc \end{pmatrix}$$

Properties of tensor product

$$\left(|a\rangle + |b\rangle\right) \otimes |c\rangle = |a\rangle \otimes |c\rangle + |b\rangle \otimes |c\rangle$$

$$|a\rangle \otimes \left(|c\rangle + |d\rangle\right) = |a\rangle \otimes |c\rangle + |a\rangle \otimes |d\rangle$$

$c(|a\rangle \otimes |c\rangle) = (c|a\rangle) \otimes |c\rangle = |a\rangle \otimes (c|c\rangle)$

$(|a\rangle \otimes |c\rangle)^{+} = |a\rangle^{+} \otimes |c\rangle^{+} \Rightarrow \langle a| \otimes \langle c|$

$(\langle a| \otimes \langle c|)(|b\rangle \otimes |d\rangle) = \langle a|b\rangle \langle c|d\rangle$

Quantum Entanglement:

$|\psi\rangle, |\phi\rangle \to \mathbb{C}^{d} \longmapsto |\psi\rangle \otimes |\varphi\rangle \to \mathbb{C}^{d^2}$

but the converse is not generally true

$\exists\ |\eta\rangle \in \mathbb{C}^{d^2}$ s.t $\nexists\ |\psi\rangle, |\phi\rangle \to \mathbb{C}^{d}$ s.t $|\psi\rangle \otimes |\varphi\rangle = |\eta\rangle$

$\hookrightarrow$ entangled states.

Schmidt decomposition $\forall\ |\eta\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, we can always

$$|\eta\rangle = \sum_{i=0}^{\min(d_1,d_2)-1} \alpha_i |a_i\rangle |b_i\rangle \leftarrow \text{schmidt vectors (orthonormal)}$$

$\llcorner$ schmidt coefficient, $\alpha_i \geq 0$

Schmidt rank: # of non-zero schmidt coefficients.

$|\eta\rangle$ is entangled iff schmidt rank > 1.

E.g: Bell Basis in $\mathbb{C}^2 \otimes \mathbb{C}^2$

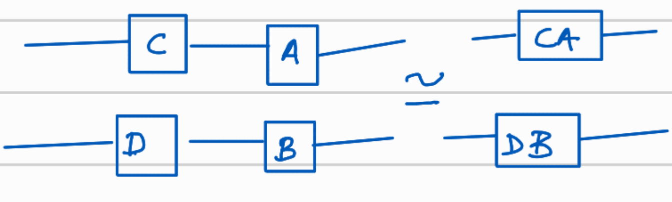
$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ $\qquad$ $|\phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$

$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ $\qquad$ $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$

Two Qubit Gates $<$ tensor product of 1-qubit gate $X \otimes Z \in U(\mathbb{C}^4)$ / genuine 2-qubit gates

i) Tensor product of 1 qubit gate

$$(\underset{q_1}{A} \otimes \underset{q_2}{B})(\underset{q_1}{C} \otimes \underset{q_2}{D}) = \underbrace{AC}_{q_1} \otimes \underbrace{BD}_{q_2}$$



$\mathrm{Tr}((X\otimes X)(X\otimes X)) = \mathrm{Tr}(X^2\otimes X^2) \Rightarrow \mathrm{Tr}(I\otimes I) \Rightarrow \mathrm{Tr}(I)\,\mathrm{Tr}(I)$

$\Rightarrow 2\times 2 = 4$

$\mathrm{Tr}(A\otimes B) = \mathrm{Tr}(A)\cdot \mathrm{Tr}(B)$

circuits. $X\otimes I$ $\quad$ $I\otimes Z$ $\quad$ $H\otimes H$

ii) Genuine 2-qubit gate $\quad U\in \mathcal{U}(\mathbb{C}^4)$

Controlled-NOT (CNOT)

$\mathrm{CNOT}|00\rangle = |00\rangle$

$\mathrm{CNOT}|01\rangle = |01\rangle$

$\mathrm{CNOT}(|10\rangle) = |11\rangle$

$\mathrm{CNOT}(|11\rangle) = |10\rangle$

$$\mathrm{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathrm{CNOT} = \begin{bmatrix} [I] & 0 \\ 0 & [X] \end{bmatrix}$$

→ Measurement.

measurement is done to extract information from the system.

→ basic type : projective / von-Neumann measurement

they are just set of projectors $B = \{\Pi_i\}_{i=0}^{m}$

$\sum_{i=0}^{m} \Pi_i = \mathbb{I}$ completeness relation

If each $\Pi_i = |\psi_i\rangle\langle\psi_i|$ i.e. rank $(\Pi_i) = 1$

say measure in the basis $\{|\psi_i\rangle\}$

Given state $|\psi\rangle \in \mathbb{C}^d$ and measurement $B = \{\Pi_i\}_{i=0}^{m} \subseteq \mathbb{C}^d$

i) $P[\text{outcome } i \,|\, |\psi\rangle) = \mathrm{Tr}\left(\Pi_i |\psi\rangle\langle\psi| \Pi_i\right)$
$= \mathrm{Tr}\left(\Pi_i^2 |\psi\rangle\langle\psi|\right) \Rightarrow \mathrm{Tr}\left(\Pi_i |\psi\rangle\langle\psi|\right)$

ii) Post measurement state

$$|\psi'\rangle = \frac{\Pi_i |\psi\rangle}{\| \Pi_i |\psi\rangle \|_2} \Rightarrow \frac{\Pi_i |\psi\rangle}{\sqrt{\langle\psi| \Pi_i |\psi\rangle}}$$

Mixed State Quantum Computation

A mixed state represented by a density operator $\rho \in L(\mathbb{C}^d)$
s.t ① $\rho_i \geq 0$ (PSD)
(ii) $\mathrm{Tr}(\rho) = 1$

quantum uncertainty - entanglement
classical uncertainty - randomness

Spectral decomposition $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

$$\sum_i p_i = 1 \text{ by (ii)}$$

Maximally mixed state $\rho \in L(\mathbb{C}^d)$ s.t $\rho = \frac{I}{d}$ - state of no information

$\forall$ orthonormal basis $\{|\psi_i\rangle\}_{i=1}^{d}$, $\sum_{i=1}^{d} |\psi_i\rangle\langle\psi_i| = I$

Partial Trace Operation

$$\mathrm{Tr}_B : L(\mathbb{C}^d_A) \times L(\mathbb{C}^d_B) \longmapsto L(\mathbb{C}^d_A)$$

tracing out the second system.

$$\mathrm{Tr}_B(\rho_{AB}) = \sum_{i=1}^{d} \left(I_A \otimes \langle i|_B\right) \rho_{AB} \left(I_A \otimes |i\rangle_B\right)$$

EXERCISE : Prove that $Tr_B(\rho_B \otimes \rho_A) = \rho_A \cdot Tr(\rho_B) = \rho_A$

Given ρ_A and ρ_B are density matrices
we know $Tr(\rho_A) = 1$ $Tr(\rho_B) = 1$

$$Tr_B(\rho_A \otimes \rho_B) = \sum_i (\rho_A \otimes \rho_B)_i$$

$$\Rightarrow \sum_i \sum_j \rho A_{ij} \rho B_{ij}$$

$$\Rightarrow \sum_i \rho A_i$$

Seperable States Pure states $|\psi\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ are not entangled, and are separable iff $|\psi\rangle = \underset{\in \mathbb{C}^{d_1}}{|\psi_1\rangle} \otimes \underset{\in \mathbb{C}^{d_2}}{|\psi_2\rangle}$

Extending this idea for mixed states.
A bipartite density matrix $\rho \in L(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ is separable if

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \otimes |\varphi_i\rangle\langle\varphi_i|$$