Quantum Control Theory

In quantum control theory, the systems to be controlled are quantum systems, whose dynamics are governed by the laws of quantum mechanics.

1) Quantum states

State of a closed quantum system can be represented by a unit vector $|\psi\rangle$ in a complex Hilbert space $\mathcal{H}$.

$$|\psi\rangle = e^{i\alpha}|\psi\rangle$$

A quantum state which can be represented with a unit vector $|\psi\rangle$ is called a pure state.

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle \qquad \theta \in [0,\pi]$$
$$\varphi \in [0, 2\pi)$$

For open quantum states and quantum ensembles we introduce density matrix $\rho : \mathcal{H} \to \mathcal{H}$ $\quad \rho^{\dagger}$ and $\mathrm{tr}(\rho) = 1$

$$\rho \equiv \sum_j p_j |\psi_j\rangle\langle\psi_j|$$

adjoint

$$\langle\psi_j| = (|\psi_j\rangle)^{\dagger}$$
$$\sum p_j = 1$$

For a pure state $|\psi\rangle$, $\rho = |\psi\rangle\langle\psi|$ and $\mathrm{tr}(\rho^2) = 1$

if $\mathrm{tr}(\rho^2) < 1$ we call it mixed state.

A composite quantum system is defined on a Hilbert space

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

$\hookrightarrow$ tensor product.

$$\rho_{AB} = \rho_A \otimes \rho_B$$

Considering any bipartite pure state $|\psi\rangle_{AB}$, if it can be written as tensor product of pure states $|\varphi\rangle_A \in \mathcal{H}_A$ $\quad |\eta\rangle_B \in \mathcal{H}_B$

$$|\psi\rangle_{AB} = |\varphi\rangle_A \otimes |\eta\rangle_B$$

we call it separable state; otherwise entangled state.

Quantum entanglement is a uniquely quantum mechanical phenomenon which plays a key role in many interested applications of quantum computation

2) Quantum Measurements

2.1) Projective measurements (Von-Neumann measurement)

To better control a quantum system, it is often desirable to extract information from the controlled quantum system, by the means of measurement.

Measurement on a quantum system unavoidably affects the measured system.

$$\{ P_m : \sum_m P_m = I, \ P_m = P_m^{+}, \ P_{\tilde{m}} P_m = \delta_{\tilde{m}m} P_m \}$$

P_m is the projector onto the eigenspace of M with an eigenvalue m,

$$M = \sum_m m P_m$$

For a quantum system in the state $|\psi\rangle$, the measurement outcome will correspond to one of the eigenvalues m of the observable M.

$$p(m) = \langle \psi | P_m | \psi \rangle$$

once the outcome 'm' has occured, the state of measured system changes to $\dfrac{P_m (|\psi\rangle)}{\sqrt{p(m)}}$ collapse postulate

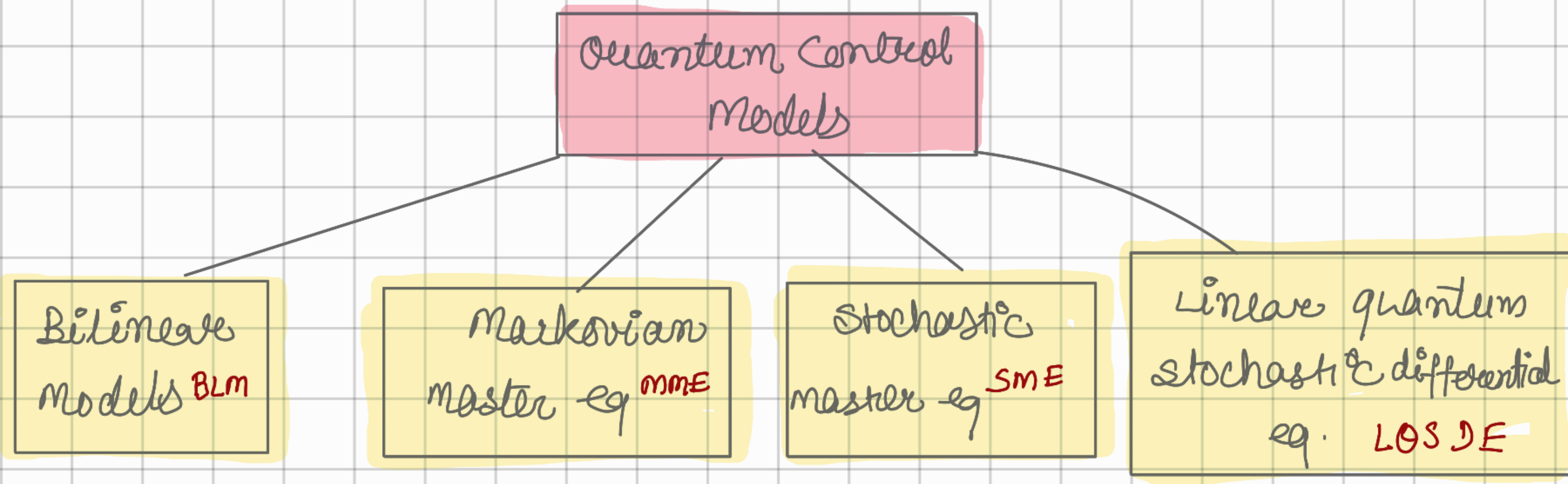
For a mixed state ρ, the probability of outcome m is $p(m) = tr\{P_m \rho\}$
and the state is changed into $\frac{P_m \rho P_m}{p(m)}$

2.2) Continuous Measurement The above framework is not very helpful to describe some important situations such as continuously monitoring some aspects of a quantum system.

In quantum feedback control it is important to continuously extract feedback information to adjust the system evolution. Continuous measurements are experimentally realizable for practical quantum systems.

In continuous measurement one can continuously monitor an observable of a quantum system and the evolution of the system described in terms of measurement records can be obtained from a stochastic master equation.

3) Quantum Control Models



3.1) Bilinear Model
The state $|\psi(t)\rangle$ of a closed quantum system evolves a/c to Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = H_0 |\psi(t)\rangle \qquad |\psi(t=0)\rangle = |\psi_0\rangle$$

free Hamiltonian

normalized

i.e $\langle \psi_0 | \psi_0 \rangle = 1$

For an N-dimensional quantum system, $\mathcal{H}$ is an N-dimensional complex Hilbert space, and the eigenstates $\{|\phi_i\rangle\}_{i=1}^{N}$ of H_0 form an orthogonal basis for $\mathcal{H}$.

In many situations control of the system may be realized by a set of control functions $u_k(t) \in \mathbb{R}$ coupled to the system via time-independent Hermitian interaction Hamiltonians H_k $(k=1,2,\dots)$

$$H(t) = H_0 + \sum_k u_k(t) H_k$$

then the controlled evolution

$$\boxed{i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \Big[H_0 + \underbrace{\sum_k u_k(t) H_k}_{\text{control}} \Big] |\psi(t)\rangle}$$

The goal in a typical quantum control problem defined on a system (as above) is to find a final time $T>0$, and a set of admissible controls $u_k(t) \in \mathbb{R}$ which drives the system from the initial state $|\psi(0)\rangle$ into a pre-defined target state $|\psi_f\rangle$.

The total Hamiltonian

$$H(t) = H_0 + \sum_k u_k(t) H_k$$

defines a unitary transformation (propagator) $U(t)$ which can accomplish the transition from a pure state $|\psi_0\rangle$ to the pure state $|\psi(t)\rangle$. i.e

$$|\psi(t)\rangle = U(t) |\psi_0\rangle$$

$$i\hbar \frac{\partial}{\partial t}\left(U(t)|\psi_0\rangle \right) = \left[H_0 + \sum_k u_k(t) H_k \right] U(t)|\psi_0\rangle$$

$$\boxed{i\hbar\, \dot{U}(t) = \left[H_0 + \sum_k u_k(t) H_k\right] U(t)}$$

Now we know that we can expand a state in terms of its eigenvalues and eigenvectors

$$|\psi(t)\rangle = \sum_{j=1}^{N} \underbrace{c_j(t)}_{\text{Eigen values}} \, \overbrace{|\phi_j\rangle}^{\text{Eigen states}}$$

$$i\hbar \frac{\partial}{\partial t}\left[\sum c_j(t)\, |\phi_j\rangle\right] = \left[H_0 + \sum_k u_k(t) H_k\right] \sum c_j(t) |\phi_j\rangle$$

$$\left[c(t) = \sum c_j(t)\right]$$

$$\boxed{i\hbar\, \dot{c}(t) = \left[H_0 + \sum_k u_k(t) H_k\right] c(t)}$$

i) $i \frac{\partial}{\partial t} |\psi(t)\rangle = \left[H_0 + \sum_k u_k(t) H_k\right] |\psi(t)\rangle$ — 5

ii) $i \dot{U}(t) = \left[H_0 + \sum_k u_k(t) H_k\right] U(t)$

$|\psi(t)\rangle = U(t)|\psi_0\rangle$ — 6

where $U(0) = I$ — 7

$|\psi(t)\rangle = \sum c_j(t) |\phi_j\rangle$ — 8

iii) $i\dot{C}(t) = \left[H_0 + \sum_k u_k(t) H_k\right] c(t)$ — 9 $\quad C_0 = (c_{0j})_{j=1}^{N}, \; c_{0j} = \langle \phi_j | \psi_0\rangle$

These three equations are all referred to as finite dimensional bilinear models (BLM) of quantum control system. If these are all controllable conversion between them is easily carried out.

Like for Eq 7 we find a set of admissible control to generate a desired unitary transformation U(t) and then calculate the trajectory using Eq. 6

3.2 Markovian Master Equations

If we make use of density matrix $\rho(t)$ to describe the state of a closed quantum system, the evolution equation for $\rho(t)$ can be described by quantum Liouville equation

$$i\dot{\rho}(t) = [H(t), \rho(t)]$$

where $[X,\rho] = X\rho - \rho X$ is the commutation operator. Most quantum systems in practical applications are open quantum systems. They unavoidably interact with their external environment. Hence it's evolution can't be unitary.

In many situations, a quantum master equation for $\rho(t)$ is a suitable way to describe the dynamics of an open quantum system. We can make use of Markovian approximation

For an N-dimensional open quantum system with Markovian dynamics, its states $\rho(t)$ can be described by the following Markovian Master Equation (MME)

$$\dot{\rho}(t) = -i[H(t), \rho(t)] + \frac{1}{2}\sum_{j,k=0}^{N^2-1} \alpha_{jk}\{[F_j\rho(t), F_k^{+}] + [F_j, \rho(t)F_k^{+}]\}$$

Here $\{F_j\}_{j=0}^{N^2-1}$ is a basis for the space of linear bounded operators on $\mathcal{H}$ with $F_0 = I$ and coeff. matrix $A = (a_{jk})$ is positive semi-definite and physically specifies the relevant relaxation rates.